Some Split Exact Sequences in the Cohomology of Groups

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Abstract

We analyze the split exact sequences of (co)homology groups associated to the spaces of Dwyer which give rise to the centralizer decomposition and subgroup decomposition of the classifying space BG of a finite group. In the first instance these sequences have infinite length. We show that they give rise to finite sequences which are also split and exact. The sequences arise as the first page of a spectral sequence.

Keywords: group cohomology, isotropy, homology decomposition, p-radical subgroup, elementary abelian subgroup.

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1. Introduction

Simplicial complexes arising from subgroups of a group have been much used in recent years in the context of group cohomology and representation theory. When p is a prime, the order complex of non-identity p-subgroups of a finite group G gives rise to split exact sequences of cohomology groups which can be used to compute the cohomology of G (see [12], [13], [14], [1], [5], [6]). This complex is also at the heart of the reformulation due to Knörr and Robinson of Alperin's weight conjecture [8]. The complex of non-identity p-subgroups was introduced by Brown [3], and later Quillen [9] and Bouc [2] introduced, respectively, the complexes of non-identity elementary abelian p-subgroups and non-identity p-radical subgroups, which turn out to have the same equivariant homotopy type as Brown's complex, and can be used in the same way.

The importance of these constructions prompts us to search for other complexes associated to G which have similar properties. This was achieved by Dwyer in [5] and [6]. Given a set of subgroups $\mathcal{C}$ of G he produced G-CW complexes $X^{\alpha}_{\mathcal{C}}$ and $X^{\beta}_{\mathcal{C}}$ which (when $\mathcal{C}$ is closed under conjugation) have the same ordinary homotopy type as the order complex of $\mathcal{C}$, but a distinct equivariant homotopy type in general. Furthermore, with suitable choices of $\mathcal{C}$, they possess the property which enables us to get split exact sequences of cohomology groups in the same way as with the order complex of $\mathcal{C}$.

In this paper we examine the split exact sequences of cohomology groups which arise from Dwyer's spaces. The problem with these spaces is that they are infinite-dimensional, and so although we obtain sequences of groups which are split and exact, because the sequences have infinite length we are unable to deduce the isomorphism type of any one term in the sequences from the remaining terms. We show that there are in fact split exact sequences of finite length constructed from the infinite sequences which allow us to deduce the cohomology of G.

In the following theorem and throughout this paper we will assume that R is a complete p-local ring, by which we mean that R is either a field of characteristic p or a complete discrete valuation ring with residue field of characteristic p. When we come to consider group homology and cohomology this includes the generality of considering the p-torsion subgroup of (co)homology of modules over arbitrary rings, since we have an isomorphism $H_u(G,M)_p \cong H_u(G,\mathbb{Z}_p\otimes_{\mathbb{Z}} M)$ where the group on the left of the isomorphism is the p-torsion subgroup and $\mathbb{Z}_p$ denotes the p-adic integers (see [12, p. 141]).

We use the notation $\mathrm{sd}_t\,\mathcal{C}$ for the set of chains $\sigma = H_0 < \cdots < H_t$ of subgroups in $\mathcal{C}$. Given such a chain σ we write $\sigma_{\mathrm{b}} = H_0$ for the bottom member of σ, and $\sigma_{\mathrm{t}} = H_t$ for the top member of σ. We write G_σ for the stabilizer of σ in the conjugation action of G, so $G_\sigma = N_G(H_0)\cap\cdots\cap N_G(H_t)$. A subgroup H of a finite group G is said to be a *p-radical subgroup* if $H = O_p(N_G(H))$, the largest normal p-subgroup of the normalizer of H.

MAIN THEOREM. *Let G be a finite group and let M be an RG-module where R is a complete p-local ring.*

(i) Let $\mathcal{C}$ be any set of non-identity p-subgroups of G which is closed under conjugation and which contains the set of non-identity elementary abelian p-subgroups of G. For each $u, s \geq 1$ there are split exact sequences

$$0 \to \bigoplus_{\sigma \in [G \backslash \operatorname{sd}_d \mathcal{C}]} H_u(C_G(\sigma_{\mathrm{t}}), M)_{G_\sigma} \to \cdots$$
$$\to \bigoplus_{\sigma \in [G \backslash \operatorname{sd}_0 \mathcal{C}]} H_u(C_G(\sigma_{\mathrm{t}}), M)_{G_\sigma} \to H_u(G, M) \to 0$$

and

$$0 \to \bigoplus_{\sigma \in [G \backslash \operatorname{sd}_d \mathcal{C}]} H_s(G_\sigma / C_G(\sigma_{\mathrm{t}}), H_u(C_G(\sigma_{\mathrm{t}}), M)) \to \cdots$$
$$\to \bigoplus_{\sigma \in [G \backslash \operatorname{sd}_0 \mathcal{C}]} H_s(G_\sigma / C_G(\sigma_{\mathrm{t}}), H_u(C_G(\sigma_{\mathrm{t}}), M)) \to 0.$$

(ii) Let $\mathcal{C}$ be the set of non-identity p-subgroups of G. For each $u, s \geq 1$ there are split exact sequences

$$0 \to \bigoplus_{\sigma \in [G \backslash \operatorname{sd}_d \mathcal{C}]} H_u(\sigma_{\mathrm{b}}, M)_{G_\sigma} \to \cdots \to \bigoplus_{\sigma \in [G \backslash \operatorname{sd}_0 \mathcal{C}]} H_u(\sigma_{\mathrm{b}}, M)_{G_\sigma} \to H_u(G, M) \to 0$$

and

$$0 \to \bigoplus_{\sigma \in [G \backslash \operatorname{sd}_d \mathcal{C}]} H_s(G_\sigma / \sigma_{\mathrm{b}}, H_u(\sigma_{\mathrm{b}}, M)) \to \cdots$$
$$\to \bigoplus_{\sigma \in [G \backslash \operatorname{sd}_0 \mathcal{C}]} H_s(G_\sigma / \sigma_{\mathrm{b}}, H_u(\sigma_{\mathrm{b}}, M)) \to 0.$$

More generally, let $\mathcal{C}$ be any set of non-identity p-subgroups of G which is closed under conjugation and which contains the set of non-identity p-radical subgroups of G. Then for such a choice of $\mathcal{C}$ the first of the sequences above is split and exact.

There are also similar sequences (with the arrows reversed) in cohomology.

We wish to point out that the first sequences in each part of this theorem have been obtained independently by Grodal [7], and that in work prior to our proof of their splitting and exactness, he proved that they are exact. We will comment more fully about this at the end of this section.

Throughout this paper we will work only with group homology, and leave it to the reader to formulate the corresponding results in group cohomology. The proofs are entirely analogous and pose no extra difficulty. It is possible to describe the component morphisms in the sequences explicitly in terms of corestriction maps in the same manner as in [14], but again we leave the details to the reader. In fact, it will be seen that the sequences depend only on the properties as a Mackey functor of $H_u(G, M)$, namely, it is projective relative to p-subgroups and vanishes when $G = 1$, and as in [14] there are also split exact sequences formulated similarly with restriction maps instead of corestriction maps.

Instead of referring to sequences as split and exact, it is convenient to say that these sequences are *contractible,* by which we mean that they are chain homotopy equivalent to the zero complex. It is an elementary exercise in the algebra of chain complexes to show that these two notions are equivalent, and the reader may also consult [13, Sect. 7].

In part (ii) of the theorem, we prove that all of the sequences are contractible only in the special case that $\mathcal{C}$ is the set of all non-identity p-subgroups of G. In the other cases where $\mathcal{C}$ is a set of p-subgroups closed under conjugation containing the p-radical subgroups we prove that the first of the sequences is contractible, and this is really the most interesting of the sequences. We have a proof in this generality that the remaining sequences are contractible, but it is rather complicated and this is not the appropriate place to record the details.

A part of this paper appears in the Ph.D. thesis of the first author [10]. There is also overlap between our results and those of Grodal [7], which were obtained independently. We would like to take the opportunity to describe the chronology of our work. The material presented here which also appears in [10] is the construction of the spectral sequence which appears at the start of Section 2 (it is described slightly differently in [10], but this does not affect the results) and the identification of the chain complexes in Proposition 2.1. In work done very soon after the Ph.D. thesis [10] was defended, the authors proved Lemma 2.2 and Proposition 2.3 which establishes the form of the sequences of the Main Theorem (although they were only written down at that time in the case of part (ii) of the theorem). In [10] it was also shown that these sequences are contractible in some very small cases. The authors were trying to prove contractibility in general when a version of [7] was received by us. In [7], Grodal obtains the sequences of the Main Theorem which have $H_u(G, M)$ as a term, and proves that they are acyclic, but by different methods to ours. We thus acknowledge Grodal's priority in proving acyclicity of these sequences. It was only after this that the authors continued their goal of proving contractibility in the generality of the Main Theorem, and came up with the argument of Section 3.

2. Dwyer's spaces, a double complex and a spectral sequence

We describe the construction of the sequences in the theorem, and start by describing the spaces of Dwyer from which they are derived. The spaces in question are the ones associated to the 'centralizer decomposition' and 'subgroup decomposition' of the classifying space BG described in [5] and [6]. The first of these may be taken to be the nerve of a category $\mathbf{X}_{\mathcal{C}}^{\alpha}$ whose objects are pairs (H, i) where H is a subgroup in $\mathcal{C}$ and $i : H \to G$ is a monomorphism from H into G with $i(H) \in \mathcal{C}$. There is a morphism $(H, i) \to (H', i')$ whenever there is a monomorphism $j : H \to H'$ so that $i = i' \circ j$. There is an action of G on this category specified by $g \cdot (H, i) = (H, c_g \circ i)$, where c_g denotes the mapping given by conjugation by $g \in G$. The stabilizer of (H, i) in G is $C_G(i(H))$.

The second space is the nerve of a category $\mathbf{X}_{\mathcal{C}}^{\beta}$ whose objects are pairs $(xH, G/H)$ where H is a subgroup in $\mathcal{C}$ and x is an element of G. There is a morphism $(xH, G/H) \to (x'H', G/H')$ whenever there is an equivariant map $\phi : G/H \to G/H'$ so that $\phi(xH) = x'H'$. The stabilizer in G of $(xH, G/H)$ is ${}^{x}H$.

Both of these categories are preordered sets, which is equivalent to saying that there is at most one morphism between each ordered pair of objects. We recall that to each preordered set there is an associated poset whose elements are the isomorphism classes of objects in the preordered set. Dwyer observes that there are G-equivariant functors $\mathbf{X}_{\mathcal{C}}^{\alpha} \to \mathcal{C}$ and $\mathbf{X}_{\mathcal{C}}^{\beta} \to \mathcal{C}$ specified, respectively, by $(H, i) \to i(H)$ and $(xH, G/H) \to {}^{x}H$. These mappings both have the property that they identify $\mathcal{C}$ as the associated poset of $\mathbf{X}_{\mathcal{C}}^{\alpha}$ and also of $\mathbf{X}_{\mathcal{C}}^{\beta}$; that is, the objects which map to a single object in $\mathcal{C}$ in every case constitute a complete isomorphism class. We do not greatly distinguish between a category and its nerve, but when we do so the notation will be that $X_{\mathcal{C}}^{\alpha}$ and $X_{\mathcal{C}}^{\beta}$ are the nerves of $\mathbf{X}_{\mathcal{C}}^{\alpha}$ and $\mathbf{X}_{\mathcal{C}}^{\beta}$. We adopt the convention used by Dwyer that the term *space* will be used to mean a simplicial set.

Associated to any G-space X and RG-module M is the so-called isotropy spectral sequence converging to the equivariant homology $H_u^G(X; M)$ ([6], see also [4]). The E^1 page of this spectral sequence consists of a lot of sequences which have the form $\mathrm{Tor}_u^{\mathbb{Z}G}(C.(X), M)$ where $C.(X)$ is the chain complex of X. We propose to call these sequences of groups also *isotropy sequences*. It is proved in [14] that if X satisfies the condition that for all non-identity p-subgroups $Q \leq G$ the fixed point space X^Q is contractible, then the augmented isotropy sequences, augmented by the morphism

$$\mathrm{Tor}_u^{\mathbb{Z}G}(C_0(X), M) \to \mathrm{Tor}_u^{\mathbb{Z}G}(\mathbb{Z}, M) = H_u(G, M),$$

are all contractible, that is, chain homotopic to the zero complex. (In [14] the condition that X be finite dimensional was not used in the proof of this theorem.) This condition is verified in many cases for the spaces $X_{\mathcal{C}}^{\alpha}$ and $X_{\mathcal{C}}^{\beta}$ by Dwyer [6], and we know from his work that in the circumstances of the Main Theorem the isotropy sequences for $X_{\mathcal{C}}^{\alpha}$ and $X_{\mathcal{C}}^{\beta}$ are contractible. The arguments which prove this can also be extracted from Section 3 of the present paper.

The problem we now face is that the isotropy sequences associated to Dwyer's spaces have infinite length, since $X^\alpha_{\mathcal{C}}$ and $X^\beta_{\mathcal{C}}$ are in general infinite dimensional. In order to deal with this we construct a spectral sequence converging to the homology of the isotropy sequence $\mathrm{Tor}^{\mathbb{Z}G}_u(C.(X), M)$ for each G-category $\mathbf{X}$. We already know in the cases of interest that this homology is $H_u(G, M)$ concentrated in degree zero. The sequences which make up the E^1 page of the spectral sequence are the sequences of the Main Theorem.

The general situation in which we construct the spectral sequence is that of a G-category $\mathbf{X}$ with the property that the composite of any two non-isomorphisms is again a non-isomorphism, a property satisfied by $\mathbf{X}^\alpha_{\mathcal{C}}$ and $\mathbf{X}^\beta_{\mathcal{C}}$. There is a direct sum decomposition of the degree n chain group $C_n(X) = \bigoplus_{r+s=n} C_{r,s}$ where $C_{r,s}$ is the span of the chains which have precisely r non-isomorphisms and s isomorphisms. Applying the boundary operator d we have $d(C_{r,s}) \subseteq C_{r-1,s} \oplus C_{r,s-1}$ and so we have a double complex whose total complex is the chain complex $C.(X)$. (We comment that we could more generally consider any partition of the morphisms of the category into two sets, $\mathcal{A}$ and $\mathcal{B}$ each closed under composition, and define $C_{r,s}$ to be the span of the chains which have precisely r morphisms in $\mathcal{A}$ and s morphisms in $\mathcal{B}$. This will give a variety of double complexes whose utility will depend on our ingenuity in choosing $\mathcal{A}$ and $\mathcal{B}$.)

Applying the functor $\mathrm{Tor}^{\mathbb{Z}G}_u(\quad, M)$ to this double complex we obtain a decomposition of each of the isotropy sequences as a double complex with terms

$$\mathrm{Tor}^{\mathbb{Z}G}_u(C_n(X), M) = \bigoplus_{r+s=n} \mathrm{Tor}^{\mathbb{Z}G}_u(C_{r,s}, M).$$

From the double complex we obtain two spectral sequences each converging to the homology of the isotropy sequence. We will consider one of these, namely the spectral sequence obtained by filtering the double complex by the subspaces spanned by the $\mathrm{Tor}^{\mathbb{Z}G}_u(C_{i,s}, M)$ with $i \le r$ for some fixed r. The rest of this section is devoted to describing the E^0 and E^1 pages of this spectral sequence.

(2.1) PROPOSITION. *The component sequences*

$$C_{r,*} = \cdots \to C_{r,2} \to C_{r,1} \to C_{r,0}$$

where $0 \le r \le \dim |\mathcal{C}|$ *have the form*

$$C_{r,*} \cong \begin{cases} \bigoplus_{\sigma \in [G \backslash \mathrm{sd}_r(\mathcal{C})]} C.(E(G_\sigma / C_G(\sigma_{\mathrm{t}}))) \uparrow^G_{G_\sigma} & \text{if } \mathbf{X} = \mathbf{X}^\alpha_{\mathcal{C}} \\ \bigoplus_{\sigma \in [G \backslash \mathrm{sd}_r(\mathcal{C})]} C.(E(G_\sigma / \sigma_{\mathrm{b}})) \uparrow^G_{G_\sigma} & \text{if } \mathbf{X} = \mathbf{X}^\beta_{\mathcal{C}}. \end{cases}$$

as complexes of $\mathbb{Z}G$ *modules. Here* $E(H)$ *is a contractible space on which* H *acts freely, for each group* H.

Proof. The two cases of the proposition are proved in the same way, and the common feature is that we have an equivariant map $f : \mathbf{X} \to \mathcal{C}$ which identifies $\mathcal{C}$ as the poset associated to the preordered set $\mathbf{X}$.

Step 1. For each chain $\sigma \in \mathrm{sd}_r(\mathcal{C})$ of length r in $\mathcal{C}$ we will write $C(\sigma)$ for the span of the chains τ in $C.(X)$ for which $f(\tau) = \sigma$. Such τ have precisely r non-isomorphisms and an arbitrary number of isomorphisms. The non-isomorphisms are mapped under f to the non-isomorphisms which make up σ. We thus have

$$C_{r,*} = \bigoplus_{\sigma \in \mathrm{sd}_r \mathcal{C}} C(\sigma)[r],$$

where $[r]$ denotes a degree shift of size r. This identification gives $C(\sigma)$ the structure of a chain complex, and its boundary map has a component zero each time a non-isomorphism is omitted from a chain. We see also that G permutes the summands in the above direct sum, and the stabilizer of the complex $C(\sigma)$ is the stabilizer G_σ of σ. Thus we may also write

$$C_{r,*} = \bigoplus_{\sigma \in [G \backslash \mathrm{sd}_r \mathcal{C}]} C(\sigma) \uparrow_{G_\sigma}^{G} [r],$$

as a sum of G-complexes.

Step 2. We prove that if $\sigma = H_0 < \cdots < H_r$ then $C(\sigma) \cong C(H_0) \otimes \cdots \otimes C(H_r)[-r]$ as complexes of G_σ-modules, where the action of G_σ on the tensor product is diagonal. We may explain this by noting that the chains τ in $C(\sigma)$ biject with $(r+1)$-tuples $(\rho_0, \ldots, \rho_r)$ of chains $\rho_i \in C(H_i)$ where each ρ_i is a chain consisting of isomorphisms which all map under f to the identity map on H_i. Here τ is obtained by splicing the chains ρ_i. These $(r+1)$-tuples form a basis for the chain complex $C(H_0) \otimes \cdots \otimes C(H_r)$, and on shifting by r we see that the isomorphism with $C(\sigma)$ given by the correspondence of basis elements just described is an isomorphism which preserves the boundary map. Since elements of G act on chains by acting on all the terms in the chains, the action of G_σ on $C(\sigma)$ is transported to the diagonal action of G_σ on the tensor product.

Step 3. We show that if H is a subgroup in $\mathcal{C}$ then

$$C(H) \cong \begin{cases} C.(E(N_G(H)/C_G(H))) & \text{if } \mathbf{X} = \mathbf{X}_{\mathcal{C}}^{\alpha} \\ C.(E(N_G(H)/H)) & \text{if } \mathbf{X} = \mathbf{X}_{\mathcal{C}}^{\beta}, \end{cases}$$

where $E(G)$ denotes a contractible space on which G acts freely. Consider the case $\mathbf{X} = \mathbf{X}_{\mathcal{C}}^{\alpha}$. An object (K, i) of $\mathbf{X}_{\mathcal{C}}^{\alpha}$ has $f(K, i) = H$ if and only if $i(K) = H$, and now $C(H)$ is the nerve of the full subcategory of $\mathbf{X}_{\mathcal{C}}^{\alpha}$ which has these (K, i) as its objects. There is precisely one morphism from any one of these objects to any other, and we see that the full subcategory with these objects is equivalent to a skeletal subcategory consisting of a single object with just the identity morphism. The nerve of the full subcategory is homotopy equivalent to the nerve of this latter category, which is contractible. The stabilizer of H as an object of $\mathcal{C}$ is $N_G(H)$, and the stabilizer of each (K, i) with $i(K) = H$ is $C_G(H)$. This shows that $N_G(H)/C_G(H)$ acts freely on the full subcategory of such (K, i), and hence the nerve is a copy of $E(N_G(H)/C_G(H))$.

The argument when $\mathbf{X} = \mathbf{X}_{\mathcal{C}}^{\beta}$ is very similar. Here the objects of $\mathbf{X}_{\mathcal{C}}^{\beta}$ which map to H are the $(xK, G/K)$ with ${}^xK = H$. There is a unique morphism from any one of these objects to any other, and the stabilizer of each such object is H. This shows that the nerve of the full subcategory of $\mathbf{X}_{\mathcal{C}}^{\beta}$ with these objects is contractible, and $N_G(H)/H$ acts freely.

Step 4. Writing $\sigma = H_0 < \cdots < H_r$, we show that

$$C(\sigma) \cong \begin{cases} C.(E(G_\sigma/C_G(H_r)))[-r] & \text{if } \mathbf{X} = \mathbf{X}_{\mathcal{C}}^{\alpha} \\ C.(E(G_\sigma/H_0))[-r] & \text{if } \mathbf{X} = \mathbf{X}_{\mathcal{C}}^{\beta}. \end{cases}$$

We have in case $\mathbf{X} = \mathbf{X}_{\mathcal{C}}^{\alpha}$,

$$\begin{aligned} C(\sigma) &\cong C(H_0) \otimes \cdots \otimes C(H_r)[-r] \\ &\cong C.(E(N_G(H_0)/C_G(H_0))) \otimes \cdots \otimes C.(E(N_G(H_r)/C_G(H_r)))[-r] \\ &\cong C.(E(N_G(H_0)/C_G(H_0)) \times \cdots \times E(N_G(H_r)/C_G(H_r)))[-r] \end{aligned}$$

by the Eilenberg-Zilber theorem. Now $E(N_G(H_0)/C_G(H_0)) \times \cdots \times E(N_G(H_r)/C_G(H_r))$ is a contractible space, since every term in the product is contractible. Since $C_G(H_0) \geq \cdots \geq C_G(H_r)$ the subgroup $C_G(H_r)$ fixes this space, and in fact $G_\sigma/C_G(H_r)$ acts freely on it, since it acts freely on the last factor. It follows that $E(N_G(H_0)/C_G(H_0)) \times \cdots \times E(N_G(H_r)/C_G(H_r))$ is G_σ-homotopy equivalent to $E(G_\sigma/C_G(H_r))$. The argument in case $\mathbf{X} = \mathbf{X}_{\mathcal{C}}^{\beta}$ is very similar. We work with $E(N_G(H_i)/H_i)$ instead of $E(N_G(H_i)/C_G(H_i))$ and use the fact that H_0 is the smallest group under consideration, instead of $C_G(H_r)$.

Step 5. We put together the previous steps to complete the proof. $\square$

We are about to identify the E^0 and E^1 pages of the spectral sequence obtained from the double complex $\{\mathrm{Tor}_u^{\mathbb{Z}G}(C_{r,s}, M)\}_{r,s}$ by filtering by the terms $\mathrm{Tor}_u^{\mathbb{Z}G}(C_{i,s}, M)$ with $i \leq r$ and no restriction on s. The calculation we use to do this is really the same as one used in identifying the terms of the Lyndon-Hochschild-Serre spectral sequence [4, p. 171].

(2.2) LEMMA. *Let N be a normal subgroup of K with factor group $Q = K/N$, let M be an RK-module, and let F be a resolution of R by projective RQ-modules. Then the complex $\mathrm{Tor}_u^{\mathbb{Z}K}(F, M)$ is isomorphic to the complex $F \otimes_{RQ} H_u(N, M)$. The degree s homology of these complexes is $H_s(Q, H_u(N, M))$.*

Proof. Let $P \to M$ be a resolution of M by projective RK-modules. Writing F_s for the term of F in degree s we have

$$\begin{aligned} \mathrm{Tor}_u^{\mathbb{Z}K}(F_s, M) &= H_u(F_s \otimes_{RK} P) \\ &= H_u(((F_s \otimes_R P)_N)_Q) \\ &= H_u((F_s \otimes_R P_N)_Q) \qquad \text{since } N \text{ acts trivially on } F_s \\ &= H_u(F_s \otimes_{RQ} P_N) \\ &= F_s \otimes_{RQ} H_u(P_N) \qquad \text{since } F_s \text{ is flat as a } RQ\text{-module} \\ &= F_s \otimes_{RQ} H_u(N, M). \end{aligned}$$

These identifications are natural in F_s, and so the complexes are isomorphic. Finally, the homology of $F \otimes_{RQ} H_u(N, M)$ is exactly the definition of $H_s(Q, H_u(N, M))$. $\square$

(2.3) PROPOSITION. *The spectral sequence obtained by filtering the double complex $\{\mathrm{Tor}_u^{\mathbb{Z}G}(C_{r,s}, M)\}_{r,s}$ by the terms $\mathrm{Tor}_u^{\mathbb{Z}G}(C_{i,s}, M)$ with $i \le r$ has E^1 page as follows.*
(i) When $\mathbf{X} = \mathbf{X}_{\mathcal{C}}^{\alpha}$ we have

$$E^1_{r,s} \cong \bigoplus_{\sigma \in [G \backslash \mathrm{sd}_r(\mathcal{C})]} H_s(G_\sigma / C_G(\sigma_{\mathrm{t}}), H_u(C_G(\sigma_{\mathrm{t}}), M)).$$

(ii) When $\mathbf{X} = \mathbf{X}_{\mathcal{C}}^{\beta}$ we have

$$E^1_{r,s} \cong \bigoplus_{\sigma \in [G \backslash \mathrm{sd}_r(\mathcal{C})]} H_s(G_\sigma / \sigma_{\mathrm{b}}, H_u(\sigma_{\mathrm{b}}, M)).$$

Proof. The E^0 term is obtained by applying $\mathrm{Tor}_u^{\mathbb{Z}G}(\ \ , M)$ to the sequences described in Proposition 2.1. We use the Eckmann-Schapiro lemma to get sequences with terms

$$\bigoplus_{\sigma \in [G \backslash \mathrm{sd}_r \mathcal{C}]} \mathrm{Tor}_u^{\mathbb{Z}G}(C.(E(G_\sigma / C_G(\sigma_{\mathrm{t}}))) \uparrow_{G_\sigma}^{G}, M) \cong \bigoplus_{\sigma \in [G \backslash \mathrm{sd}_r \mathcal{C}]} \mathrm{Tor}_u^{\mathbb{Z}G_\sigma}(C.(E(G_\sigma / C_G(\sigma_{\mathrm{t}}))), M)$$

in case $\mathbf{X} = \mathbf{X}_{\mathcal{C}}^{\alpha}$. Taking $K = G_\sigma$ and $N = C_G(\sigma_{\mathrm{t}})$ we may apply the last lemma, since $C.(E(G_\sigma / C_G(\sigma_{\mathrm{t}})))$ is a resolution of R by projective $R[G_\sigma / C_G(\sigma_{\mathrm{t}})]$-modules. The degree s homology is exactly as stated in the proposition. The appropriate expressions when $\mathbf{X} = \mathbf{X}_{\mathcal{C}}^{\beta}$ are obtained similarly. $\square$

The E^1 page is in fact made up of a lot of sequences with terms as just described, and these are the sequences of the Main Theorem, augmented by the term $H_u(G, M)$ in the case of the sequence at $s = 0$.

3. Contractibility of the sequences

In this section we prove the contractibility of the sequences of the Main Theorem. We have just seen that these sequences make up the E^1 page of a spectral sequence which converges to the homology of an isotropy sequence $\mathrm{Tor}_u^{\mathbb{Z}G}(C.(X), M)$. In the cases of interest in the Main Theorem, this isotropy sequence is contractible. We will see that this contraction can be given by a chain homotopy equivalence of a certain form, which we describe in the next lemma, and it is this property which enables us to deduce the contractibility of the sequences in the Main Theorem.

(3.1) LEMMA. *Let $D_{r,s}$ be a first quadrant double complex with a map $T : D \to D$ of total degree $+1$ so that $Td + dT = 1$ and $T(D_{r,s}) \subseteq \bigoplus_{\substack{a+b=r+s+1\\ a\leq r+1}} D_{a,b}$. Let us write $d = d_1 + d_2$ where $d_1 : D_{r,s} \to D_{r-1,s}$ and $d_2 : D_{r,s} \to D_{r,s-1}$. Then on the homology $H(d_2)$ of d_2 there is induced a map $T_1 : H(d_2) \to H(d_2)$ of degree $+1$ such that $T_1 d_1 + d_1 T_1 = 1$.*

The crucial hypothesis in this lemma is that $a \leq r + 1$. We comment that this is automatically satisfied if $s = 0$, which corresponds to considering the terms along an edge of the double complex. When we come to apply this lemma to prove the contractibility of the sequences in the main theorem the sequences along the edge are the ones which have the term $H_u(G, M)$ in them.

Proof. The idea is simply that a contracting homotopy of a filtered complex which raises the filtration degree by one also induces a contracting homotopy on the filtration quotients. We spell out the details of this idea.

We may write $T = \sum_{i\geq 0} f_i$ where $f_i : D_{r,s} \to D_{r+1-i,s+i}$ and now the condition $Td + dT = 1$ implies the equations

$$f_0 d_2 + d_2 f_0 = 0$$
$$f_0 d_1 + d_1 f_0 + f_1 d_2 + d_2 f_1 = 1$$

as well as some other equations which we do not write down. If now $z \in D_{r,s}$ with $d_2(z) = 0$ then the first equation implies that

$$d_2 f_0(z) = -f_0 d_2(z) = 0,$$

so that $f_0(z)$ is a cycle. Also if $z \in D_{r,s}$ has the form $z = d_2(w)$ for some $w \in D_{r,s+1}$ then the first equation shows that

$$f_0(z) = f_0 d_2(w) = -d_2 f_0(w)$$

is a boundary. This shows that the component f_0 induces a map on homology $T_1 : H(d_2) \to H(d_2)$.

We now show that $T_1d_1 + d_1T_1 = 1$ on $H(d_2)$. This follows from the second equation, which shows that on $D_{r,s}$ we have

$$f_0d_1 + d_1f_0 - 1 = -f_1d_2 - d_2f_1.$$

If z is a cycle then

$$(-f_1d_2 - d_2f_1)(z) = -d_2f_1(z)$$

which is zero in homology, so we deduce that $f_0d_1 + d_1f_0 - 1$ induces zero on $H(d_2)$. $\square$

We will apply this lemma to the double complex whose terms are $\mathrm{Tor}_u^{\mathbb{Z}G}(C_{r,s}, M)$, augmented by introducing a further term $H_u(G, M) = \mathrm{Tor}_u^{\mathbb{Z}G}(\mathbb{Z}, M)$ in bidegree $(-1, 0)$, with the differential component $d_1 : \mathrm{Tor}_u^{\mathbb{Z}G}(C_{0,0}, M) \to H_u(G, M)$ being induced by the augmentation $C_{0,0} \to \mathbb{Z}$. The total complex of this augmented double complex is now the augmented isotropy sequence associated to the space X. We will show that when X is one of $X_{\mathcal{C}}^{\alpha}$ or $X_{\mathcal{C}}^{\beta}$ where $\mathcal{C}$ is one of the possibilities mentioned in the Main Theorem, the augmented isotropy sequence is chain homotopy equivalent to the zero complex by a chain homotopy satisfying the condition of Lemma 3.1. This will show that the sequences in the Main Theorem are contractible.

Proof that the sequences of the Main Theorem are contractible. Ultimately we will consider the cases $X_{\mathcal{C}}^{\alpha}$ and $X_{\mathcal{C}}^{\beta}$ separately, but the initial part of the argument works in all cases.

Step 1: reduction to p-subgroups. Let P be a Sylow p-subgroup of G. The restriction map $\mathrm{Tor}_u^{\mathbb{Z}G}(C_{r,s}, M) \to \mathrm{Tor}_u^{\mathbb{Z}P}(C_{r,s}, M)$ is a natural transformation of cohomological functors which is naturally split by a scalar multiple of corestriction, and so these mappings express the double complex $\mathrm{Tor}_u^{\mathbb{Z}G}(C_{r,s}, M)$ as a direct summand of the double complex $\mathrm{Tor}_u^{\mathbb{Z}P}(C_{r,s}, M)$. It follows that the homology with respect to d_2 of the first complex is a direct summand of the homology with respect to d_2 of the second complex, and since direct summands of contractible complexes are contractible, it suffices to show that the sequences which make up the homology with respect to d_2 of the $\mathrm{Tor}_u^{\mathbb{Z}P}(C_{r,s}, M)$ are contractible.

Step 2: removal of free orbits. Since $\mathrm{Tor}_u^{\mathbb{Z}P}(C_{r,s}, M)$ vanishes on the span in $C_{r,s}$ of the chains in free orbits under the action of P, we may replace X by the subspace consisting of the simplices where the action of P is not free, and this is $\bigcup_{1\neq Q\leq P} X^Q$. It is the nerve of the subcategory $\bigcup_{1\neq Q\leq P} \mathbf{X}^Q$ where $\mathbf{X}^Q$ is the subcategory of $\mathbf{X}$ whose objects and morphisms are fixed by Q.

Step 3: analysis of the contracting homotopy. It is known, as in [6] and [11], that $\bigcup_{1\neq Q\leq P} \mathbf{X}^Q$ is contractible equivariantly for P, and that the contraction may be given by a sequence of functors of the form described in the next lemma and corollary. These results allow us to show that the hypothesis of Lemma 3.1 is satisfied.

(3.2) LEMMA. *Let $F_1, F_2 : \mathbf{X} \to \mathbf{X}$ be functors with either a natural transformation $\eta : F_1 \to F_2$ or a natural transformation $\eta : F_2 \to F_1$. Then there is a map $T : C.(X) \to C.(X)$ of degree $+1$ which is induced by η and satisfies $Td+dT = F_2 - F_1$ with the property that for each chain γ, $T(\gamma)$ is a sum of terms each of which only has at most one more non-isomorphism in it than γ.*

Proof. Assume that $\eta : F_1 \to F_2$, the other case being treated by interchanging F_1 and F_2 and replacing T by $-T$. If $\gamma = x_0 \to x_1 \to \cdots \to x_r$ the construction of $T(\gamma)$ is that it is the alternating sum

$$T(\gamma) = \sum_{i=0}^{r} (-1)^i F_1(x_0) \to \cdots \to F_1(x_i) \to F_2(x_i) \to \cdots \to F_2(x_r).$$

If any $x_{i-1} \to x_i$ is an isomorphism then so are $F_1(x_{i-1}) \to F_1(x_i)$ and $F_2(x_{i-1}) \to F_2(x_i)$, so that the morphisms other than $F_1(x_i) \to F_2(x_i)$ in each summand of $T(\gamma)$ account for at most as many non-isomorphisms as there were in γ. The extra morphism $F_1(x_i) \to F_2(x_i)$ introduces at most one more. $\square$

We comment that the condition that each term of $T(\gamma)$ has at most one more non-isomorphism than γ is the link with Lemma 3.1 in our particular application, since it guarantees the condition $a \le r+1$ which appears in Lemma 3.1. We extend Lemma 3.2 to apply to a sequence of functors linked by natural transformations.

(3.3) COROLLARY. *Suppose we have functors $F_1, \ldots, F_a$ such that for each i there is either a natural transformation $\eta_i : F_i \to F_{i+1}$ or a natural transformation $\eta_i : F_{i+1} \to F_i$. Then there is a mapping $T : C.(X) \to C.(X)$ of degree $+1$ which is induced by the η_i and satisfies $Td+dT = F_a - F_1$ with the property that for each chain γ, $T(\gamma)$ is a sum of terms each of which only has at most one more non-isomorphism in it than γ.*

Proof. For each i by Lemma 3.2 we find T_i so that $T_i d + dT_i = F_{i+1} - F_i$ and each summand of $T_i(\gamma)$ has at most one more non-isomorphism than γ. Now putting $T = T_1 + \cdots + T_{a-1}$ we have $Td + dT = F_a - F_1$ and still the summands of $T(\gamma)$ have at most one more non-isomorphism than γ. $\square$

We now show for the particular choices of $\mathbf{X}$ in the Main Theorem that the space $\bigcup_{1 \ne Q \le P} \mathbf{X}^Q$ is contractible by a chain of functors as in Corollary 3.3, which are also equivariant for P. When $\mathbf{X} = \mathbf{X}_{\mathcal{C}}^{\alpha}$ and $\mathcal{C}$ is any set of non-identity p-subgroups of G which contains the non-identity elementary abelian p-subgroups of G, we contract $\mathbf{Y} = \bigcup_{1 \ne Q \le P} (\mathbf{X}_{\mathcal{C}}^{\alpha})^Q$ by the functors $F_1, F_2, F_3 : \mathbf{Y} \to \mathbf{Y}$ such that F_1 is the identity,

$$F_2(H, i) = (i(H) \cdot \Omega_1 Z(P), \text{inclusion})$$

and

$$F_3(H,i) = (\Omega_1 Z(P), \text{inclusion}),$$

where $\Omega_1 Z(P)$ denotes the largest elementary abelian subgroup of the centre of P. Thus F_3 is a constant functor. There are natural transformations $\eta_1 : F_1 \to F_2$ and $\eta_2 : F_3 \to F_2$ specified at each object as the unique possible morphism.

When $\mathbf{X} = \mathbf{X}^\beta_{\mathcal{C}}$ and $\mathcal{C}$ is the set of all non-identity p-subgroups of G we contract $\mathbf{Y} = \bigcup_{1 \neq Q \leq P} (\mathbf{X}^\beta_{\mathcal{C}})^Q$ by the functors $F_1, F_2, F_3 : \mathbf{Y} \to \mathbf{Y}$ such that F_1 is the identity,

$$F_2(xH, G/H) = (P \cap {}^xH, G/(P \cap {}^xH))$$

and

$$F_3(xH, G/H) = (P, G/P).$$

There are natural transformations $\eta_1 : F_2 \to F_1$ and $\eta_2 : F_2 \to F_3$ specified at each object as the unique possible morphism. We note also that if $(xH, G/H)$ is fixed by Q then $Q \subseteq {}^xH$ so that $P \cap {}^xH \neq 1$.

The remaining case of $\mathbf{X} = \mathbf{X}^\beta_{\mathcal{C}}$ where $\mathcal{C}$ is a subset of the non-identity p-subgroups of G containing the p-radical subgroups is more complicated. It is known from [11] that $\mathbf{X} = \mathbf{X}^\beta_{\mathcal{C}}$ is equivariantly homotopy equivalent to the corresponding space for the set of all non-identity p-groups, which enables us to deduce that $\bigcup_{1 \neq Q \leq P} (\mathbf{X}^\beta_{\mathcal{C}})^Q$ is equivariantly contractible for the action of P since it is also true for the case of all non-identity p-subgroups and this property is preserved under the equivariant homotopy equivalence. The comment immediately after the statement of Lemma 3.1 now allows us to deduce without further effort that the sequence of the Main Theorem which contains $H_u(G, M)$ is contractible in this case.

The ingredients in the proof of the Main Theorem are now all in place, and we summarize the argument. The sequences of the Main Theorem are obtained by Proposition 2.3 as the homology of a double complex with respect to the differential in one direction, in the manner of Lemma 3.1. To show these sequences are contractible it suffices to consider the action of a Sylow p-subgroup P of G on a subspace $\bigcup_{1 \neq Q \leq P} X^Q$. We have just verified that this subspace has a P-equivariant contraction of the kind required as a hypothesis for Lemma 3.1. Finally by Lemma 3.1 we obtain the contractibility of the sequences of the Main Theorem. $\square$

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