# STANDARD STRATIFICATIONS OF EI CATEGORIES AND ALPERIN'S WEIGHT CONJECTURE

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Abstract. We characterize the finite EI categories whose representations are standardly stratified with respect to the natural preorder on the simple representations. The orbit category of a finite group with respect to any set of subgroups is always such a category. Taking the subgroups to be the $p$-subgroups of the group, we reformulate Alperin's weight conjecture in terms of the standard and proper costandard representations of the orbit category. We do this using the properties of the Ringel dual construction and a theorem of Dlab, which have elsewhere been described for standardly stratified algebras where there is a partial order on the simple modules. We indicate that these results hold in the generality of an algebra whose simple modules are preordered, rather than partially ordered.

## 1. Introduction

An EI category is one in which every endomorphism is an isomorphism. In this paper we are particularly motivated by the examples of EI categories which are constructed from families of subgroups of a given group. These include the orbit category of a group with respect to a family of subgroups, the Frobenius category, the transporter category, and the Brauer category of a block. Groups themselves, as well as partially-ordered sets are also examples of EI categories.

A representation of a category over a commutative ring is simply a functor from the category to the category of modules for that ring. When the category is a group this is a representation in the usual sense, and if the category happens to be a poset we obtain a representation of the incidence algebra of the poset. We develop a particular aspect of the representation theory of EI categories, and this is a combination of the representation theories of groups and of posets.

The main application we give in this paper of the theory of representations of EI categories is to the local determination of properties of representations of finite groups, and in particular to Alperin's Weight

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Conjecture. We reformulate this conjecture in terms of the standard and proper costandard modules for a standardly stratified algebra. A module category which is standardly stratified has similar properties to a highest weight category, but the notion is more general, and the aim in considering such structure is to simulate aspects of weight theory, such as appears with the representations of a complex semisimple Lie algebra. Before reaching the application to Alperin's conjecture we first characterize the EI categories whose representations are standardly stratified with respect to a natural preorder on the simple modules. We do this in Section 2. It turns out that representations of the orbit category of a finite group with respect to a family of subgroups are always standardly stratified, and the stratification is related to the procedure used to compute the derived functors of the limit functor in [15] and [22].

In order to present our reformulations of Alperin's conjecture, we need to invoke the development of the theory of standardly stratified algebras in a broader context than the one which appears in many accounts of the subject. The definition of a standardly stratified algebra given by Cline, Parshall and Scott [8] has as one of its ingredients a preordered set which parametrizes the simple modules. The key results which we will invoke are a theorem of Dlab and the properties of the Ringel dual algebra, and in [1, 2, 10, 17, 30] these were established under the hypothesis that the preordered set is actually a partially-ordered set. In [29] the Ringel dual algebra was introduced using a preordered set, and it was shown there that the dual algebra is again standardly-stratified, but it was not shown that a standardly-stratified algebra may be identified with its double dual, and neither was Dlab's theorem considered. In fact all of these results hold in the generality that the simple modules are parametrized by a preordered set provided we are careful to set things up in the right way. Our purpose in Section 3 of this paper is to show how to do this. We have been informed since writing this paper that such a development has also been given in [13], which appeared in preprint form a few months earlier than the present paper.

In the last section of this paper we provide our reformulations of Alperin's weight conjecture in the context of stratifications. We work with the representations of the orbit category of a finite group $G$ with respect to its $p$-subgroups for some prime $p$ and we consider the standard representations $\Delta_\lambda$, the proper costandard representations $\overline{\nabla}_\lambda$ and the canonical or partial tilting representations $T_\lambda$ over an algebraically closed field of characteristic $p$. We show that Alperin's weights correspond to the $\Delta_\lambda$ which are simple, and also to the $\overline{\nabla}_\lambda$ which are injective. These in turn correspond by duality to the $\overline{\nabla}_\lambda$ for the Ringel dual algebra which are $T_\lambda$. At the same time the simple representations of $G$ are in correspondence with the $\Delta_\lambda$ which are $T_\lambda$, the $\Delta_\lambda$ which are injective, and also the $\Delta_\lambda$ for the Ringel dual algebra which are projective. Thus various reformulations of the conjecture that the number of weights equals the number of simple modules for $G$ become apparent, and certain of these reformulations possess a symmetry which is lacking in some other forms of the conjecture. Thus Alperin's weight conjecture is equivalent to the statement that the number of $\overline{\nabla}_\lambda$ for the orbit category which are injective equals the number of $\Delta_\lambda$ for the Ringel dual algebra which are projective; and it is also equivalent to the statement that the number of $\Delta_\lambda$ equal to $T_\lambda$ for the the orbit category equals the number of $\overline{\nabla}_\lambda$ equal to $T_\lambda$ for the Ringel dual algebra.

## 2. Some EI category algebras are standardly stratified

We start by describing some basic properties of representations of categories, and of EI categories in particular. Given a small category $\mathcal{C}$ and a field $k$ (which could more generally be a commutative ring $R$ with a 1), we may form the *category algebra* $k\mathcal{C}$ which is the $k$-vector space with the morphisms of $\mathcal{C}$ as a basis. A multiplication is defined on the basis elements by

$$\alpha\beta = \begin{cases} \alpha \circ \beta & \text{if } \alpha \text{ and } \beta \text{ can be composed,} \\ 0 & \text{otherwise.} \end{cases}$$

We will compose morphisms on the left and deal with left modules, so that here $\alpha \circ \beta$ means do $\beta$ first, then $\alpha$. This is a construction which generalizes the notion of the group algebra of a group and incidence algebra of a poset, and the multiplication makes $k\mathcal{C}$ into an associative algebra, which has an identity element if and only if $\mathcal{C}$ has finitely many objects. In that case the identity is $1 = \sum_{x \in \mathrm{Ob}\,\mathcal{C}} 1_x$ and the elements $1_x$ form a set of pairwise mutually orthogonal idempotents in $k\mathcal{C}$.

A *representation* of a category $\mathcal{C}$ is a functor $F : \mathcal{C} \to k$-mod. Extending the familiar property of the group algebra when $\mathcal{C}$ happens to be a group, we immediately see that representations of $\mathcal{C}$ may be identified with $k\mathcal{C}$-modules, in that given a functor $F : \mathcal{C} \to k$-mod we obtain a $k\mathcal{C}$-module $\bigoplus_{x \in \mathrm{Ob}\,\mathcal{C}} F(x)$, and given a $k\mathcal{C}$-module $M$ we obtain a functor whose value at $x \in \mathcal{C}$ is $1_x \cdot M$. Evidently natural transformations of functors correspond to module homomorphisms under this identification.

We start with two general results, the first of which shows that in considering the representations of a category $\mathcal{C}$, we can replace $\mathcal{C}$ by any full subcategory which contains at least one object from each isomorphism class.

**Lemma 2.1.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be equivalent categories. Then the categories of representations of $\mathcal{C}$ and of $\mathcal{C}'$ are also equivalent.*

*Proof.* The equivalence of categories means there are functors $A : \mathcal{C} \to \mathcal{C}'$ and $B : \mathcal{C}' \to \mathcal{C}$ so that $AB$ and $BA$ are naturally isomorphic to the identity functors. Now if $F : \mathcal{C} \to k$-mod is a representation of $\mathcal{C}$ we obtain a representation $FB$ of $\mathcal{C}'$ and if $G$ is a representation of $\mathcal{C}'$ we obtain a representation $GA$ of $\mathcal{C}$. On composing the natural isomorphism of $BA$ and the identity with $F$ we obtain a natural isomorphism between the functor $F \mapsto FBA$ and the identity functor $F \mapsto F$, and similarly the functor $G \mapsto GAB$ is isomorphic to $G \mapsto G$. This demonstrates the equivalence of categories of representations. $\square$

Our next result shows that the category algebras of categories with more than one isomorphism class of objects of the kind we will consider in the remainder of this paper are always non-trivially stratified in the sense of [8] . We need to know the definition of a stratifying ideal $J$ in a finite-dimensional algebra A over a field, and this may be given in more than one way. One definition is that it is an ideal for which the inflation map induces an isomorphism $\operatorname{Ext}^*_{A/J}(U, V) \cong \operatorname{Ext}^*_A(U, V)$ for all $A/J$-modules $U$ and $V$. In the proof of our result we find it easier to work with an equivalent definition which appears in [8].

**Proposition 2.2.** *Let $\mathcal{D}$ be a full subcategory of a finite category $\mathcal{C}$, and let $e = \sum_{x\in\mathcal{D}} 1_x \in k\mathcal{C}$ where $k$ is a field. Let $\mathcal{E}$ be the full subcategory of $\mathcal{C}$ whose objects are the objects of $\mathcal{C}$ not in $\mathcal{D}$, and suppose that for all $x \in \mathcal{D}$ and $y \in \mathcal{E}$ we have $\operatorname{Hom}(x, y) = \emptyset$. Then*

(1) *$J = k\mathcal{C}ek\mathcal{C}$ is a stratifying ideal in $k\mathcal{C}$, and*
(2) *for all $k\mathcal{E}$-modules $U$ and $V$ we have $\operatorname{Ext}^*_{k\mathcal{E}}(U, V) \cong \operatorname{Ext}^*_{k\mathcal{C}}(U, V)$, where $U$ and $V$ are regarded as $k\mathcal{C}$-modules by inflation.*

*Proof.* (1) We verify that the conditions of Definition 2.1.1 of [8] are satisfied. By construction $J$ is idempotent, and we must verify that multiplication induces an isomorphism $k\mathcal{C}e \otimes_{ek\mathcal{C}e} ek\mathcal{C} \to J$ and

$$\operatorname{Tor}_n^{ek\mathcal{C}e}(k\mathcal{C}e, ek\mathcal{C}) = 0 \text{ for all } n > 0.$$

Now $k\mathcal{C}e = ek\mathcal{C}e$ since both sides are the span in $k\mathcal{C}$ of all morphisms in $\mathcal{D}$. This immediately implies that the Tor groups are zero because

$k\mathcal{C}e$ is projective as an $ek\mathcal{C}e$-module. We also have an isomorphism

$$\begin{aligned} k\mathcal{C}e \otimes_{ek\mathcal{C}e} ek\mathcal{C} &= ek\mathcal{C}e \otimes_{ek\mathcal{C}e} ek\mathcal{C} \\ &\cong ek\mathcal{C} \\ &= k\mathcal{C}ek\mathcal{C} \end{aligned}$$

induced by multiplication, the latter equality holding since both sides are the span in $k\mathcal{C}$ of the morphisms in $\mathcal{C}$ which have target in $\mathcal{D}$.

(2) We may regard $U$ and $V$ as $k\mathcal{C}$-modules via the homomorphism $k\mathcal{C} \to k\mathcal{E}$ which sends morphisms not in $\mathcal{E}$ to zero. Now the statement is equivalent to (1), as described in [8, 2.1.2]. □

If $\mathcal{C}$ is an EI category, that is, one in which all endomorphisms are isomorphisms, we see immediately that the set of isomorphism classes $[x]$ of objects $x$ in $\mathcal{C}$ forms a poset under the relation $[x] \leq [y]$ if and only if there is a morphism $x \to y$. Thus if $\mathcal{D}$ is any full subcategory with the property that $x \in \mathcal{D}$ and $[x] \leq [y]$ imply $y \in \mathcal{D}$, then $\mathcal{D}$ satisfies the hypothesis of the proposition. We may consider a chain of such subcategories $\emptyset = \mathcal{D}_0 \subset \cdots \subset \mathcal{D}_n = \mathcal{C}$ in which each subcategory has one more isomorphism class of objects than the previous one, so that $n$ is the number of isomorphism classes in $\mathcal{C}$, and from this we obtain a stratification of $k\mathcal{C}$ of length $n$. For each $k\mathcal{C}$-module $M$ there is a corresponding filtration in which each factor is zero except on one isomorphism class of objects, where it is the value of $M$. This is exactly the filtration which has been used in the calculation of higher limits in [15] and [22].

Whereas category algebras of finite EI categories are always stratified, with a stratification of length equal to the number of isomorphism classes in the category, they are not always standardly stratified, and in the remainder of this section we characterize the circumstances in which this happens. We first recall the definition. Suppose that $A$ is a finite-dimensional $k$-algebra whose simple modules are parametrized as $S_\lambda$, $\lambda \in \Lambda$ where $\Lambda$ is a preordered set (namely a set with a reflexive, transitive relation $\leq$). We write $\lambda < \mu$ (where $\lambda, \mu \in \Lambda$) to mean $\lambda \leq \mu$ but $\mu \not\leq \lambda$. Let $P_\lambda$ be the projective cover of $S_\lambda$. According to [8, 2.2.1], the algebra $A$ is *standardly-stratified* with respect to $(\Lambda, \leq)$ if there exist modules $\Delta_\lambda$, $\lambda \in \Lambda$, with the following properties:

(1) if the composition factor multiplicity $[\Delta_\lambda : S_\mu] \neq 0$ then $\mu \leq \lambda$, and
(2) for each $\lambda \in \Lambda$ there is a surjection $P_\lambda \to \Delta_\lambda$ so that the kernel $K_\lambda$ has a filtration with factors $\Delta_\mu$ where $\mu > \lambda$.

We readily see that if $A$ is standardly-stratified then the $\Delta_\lambda$ are determined as the modules $P_\lambda/K_\lambda$ where

$$K_\lambda = \sum_{\substack{\phi: P_\mu \to P_\lambda \\ \mu > \lambda}} \phi(P_\mu)$$

is the trace in $P_\lambda$ of the projective modules $P_\mu$ with $\mu > \lambda$.

We next recall the parametrization of the simple and projective representations of an EI category, which may be found in [9] and [20], and we summarize this description. The simple representations of an EI category $\mathcal{C}$ are in bijection with pairs $(x, V)$ where $x$ is an object of $\mathcal{C}$ taken up to isomorphism, and $V$ is a simple $k\,\mathrm{Aut}(x)$-module, again taken up to isomorphism. Indeed, given a pair $(x, V)$ one readily constructs a simple functor whose support is the isomorphism class $[x]$ of $x$ and whose value on an object $x'$ isomorphic to $x$ is $V$ with the action of $\mathrm{Aut}(x')$ on $V$ obtained by transporting the action of $\mathrm{Aut}\,x$ via some fixed isomorphism $x \to x'$. (Alternatively we may replace $\mathcal{C}$ by a skeletal subcategory by virtue of Lemma 2.1, in which case the simple functor associated to $(x, V)$ is zero except on $x$, where it is $V$.) We will denote the simple functor so constructed by $S_{x,V}$. An elementary argument shows that these simple representations form a complete list of isomorphism types.

In the next result we describe the projective cover $P_{x,V}$ of $S_{x,V}$.

**Proposition 2.3.** [9] *Let $\mathcal{C}$ be a finite EI category and $k$ a field. The projective cover $P_{x,V}$ of $S_{x,V}$ is a direct summand of the functor $k[\mathrm{Hom}(x, -)]$, which is projective. It is supported on $\{y \in \mathrm{Ob}\,\mathcal{C} \mid$ there is a morphism $x \to y\}$. On evaluation at $x$, $P_{x,V}(x)$ is the projective cover of $V$ as a $k\,\mathrm{Aut}(x)$-module.*

*Proof.* Since $1 = \sum_{x \in \mathrm{Ob}\,\mathcal{C}} 1_x$ is a sum of mutually orthogonal idempotents, $k\mathcal{C} \cdot 1_x$ is a projective module for $k\mathcal{C}$, and regarded as a functor this is $k[\mathrm{Hom}(x, -)]$ since on evaluation at an object $y$ this is a vector space with the morphisms $x \to y$ as a basis, and the same is true of $1_y \cdot k\mathcal{C} \cdot 1_x$. Let $1_x = e + f$ be a sum of orthogonal idempotents in $k\,\mathrm{Aut}(x)$ where $k\,\mathrm{Aut}(x) \cdot e$ is the projective cover of $V$. Then $k\mathcal{C}e$ is a summand of $k[\mathrm{Hom}(x, -)]$, hence is projective, and it is indecomposable since it is generated by its value at $x$, where it is an indecomposable $k\,\mathrm{Aut}(x)$-module, namely the projective cover of $V$. Since $eS_{x,V} \neq 0$ we have constructed $P_{x,V}$. Its support is contained in the support of $k[\mathrm{Hom}(x, -)]$, which equals $\{y \in \mathrm{Ob}\,\mathcal{C} \mid$ there is a morphism $x \to y\}$. $\square$

When $A = k\mathcal{C}$ is the category algebra of an EI category $\mathcal{C}$, we take $\Lambda$ to be the set of all pairs $(x, V)$ where $x \in \operatorname{Ob}\mathcal{C}$ and $V$ is a simple $k\operatorname{Aut}(x)$-module, taken up to isomorphism. There is a *canonical preorder* on $\Lambda$ given by $(x, V) \leq (y, W)$ if and only if there exists a morphism $x \to y$ in $\mathcal{C}$. In the remainder of this section we characterize the EI categories $\mathcal{C}$ for which $k\mathcal{C}$ is standardly stratified with respect to this preorder. For each $\lambda = (x, V)$ we now define $A$-modules $K_\lambda$ as above. Explicitly:

$$K_{x,V} = \sum_{\substack{\phi : P_{y,W} \to P_{x,V} \\ (y,W) > (x,V)}} \phi(P_{y,W}).$$

We define $\Delta_{x,V} = P_{x,V}/K_{x,V}$.

**Lemma 2.4.** *Let $\mathcal{C}$ be a finite EI category. Then*

$$K_{x,V}(y) = \begin{cases} P_{x,V}(y) & \text{if there is a non-isomorphism } x \to y, \\ 0 & \text{otherwise,} \end{cases}$$

*and*

$$\Delta_{x,V}(y) = \begin{cases} P_V & \text{if } x = y, \\ 0 & \text{if } x \not\cong y \end{cases}$$

*where $P_V$ is the projective cover of $V$ as an $k\operatorname{Aut}(x)$-module.*

*Proof.* The elements of $\Lambda$ strictly greater than $(x, V)$ are the $(y, W)$ for which there is a non-isomorphism $x \to y$. The corresponding indecomposable projective $k\mathcal{C}$-modules $P_{y,W}$ are precisely the indecomposable summands of the modules $k\mathcal{C} \cdot 1_y$, by Proposition 2.3. Thus the sum of images in $P_{x,V}$ of homomorphisms from the $P_{y,W}$ with $(x, V) < (y, W)$ equals the sum of the images of the homomorphisms from the $k\mathcal{C} \cdot 1_y$ for which there is a non-isomorphism $x \to y$, and evaluated at such $y$ this is $P_{x,V}(y)$. Since $k\mathcal{C} \cdot 1_y$ only has support on objects $z$ for which there is a non-isomorphism $y \to z$, it follows that $K_{x,V}(y) = 0$ if there is no non-isomorphism $x \to y$. From Proposition 2.3 it now follows that $\Delta_{x,V}$ only has support on the isomorphism class of $x$, where it is $P_V$. ∎

We come now to the main result of this section.

**Theorem 2.5.** *Let $\mathcal{C}$ be a finite EI category and let $k$ be a field. Let $\Lambda$ be the preordered set of pairs $(x, V)$ which parametrizes the set of isomorphism classes of simple modules of the category algebra $k\mathcal{C}$. Then $k\mathcal{C}$ is standardly stratified with respect to $\Lambda$ if and only if for every morphism $\alpha : x \to y$ in $\mathcal{C}$ the group $\operatorname{Stab}_{\operatorname{Aut}(y)}(\alpha) = \{\theta \in \operatorname{Aut}(y) \mid \theta\alpha = \alpha\}$ has order which is invertible in $k$.*

*Proof.* We know that if $k\mathcal{C}$ is standardly stratified then the standard modules are the $\Delta_{x,V}$ defined earlier. Furthermore, the composition factors of $\Delta_{x,V}$ are precisely the $S_{x,W}$ where $W$ is a composition factor of $P_V$, and these always satisfy $(x,W) \leq (x,V)$. From this we see that $k\mathcal{C}$ is standardly stratified if and only if each module $K_{x,V}$ has a filtration with factors of the form $\Delta_{y,W}$ where $y > x$.

We claim that $K_{x,V}$ has a filtration with factors $\Delta_{y,W}$ where $y > x$ if and only if each evaluation $K_{x,V}(y)$ is projective as an $k\,\mathrm{Aut}(y)$-module. This is because $\Delta_{y,W}$ is supported only on (the isomorphism class of) $y$, where it is the projective module $P_W$, and $K_{x,V}$ is supported only on objects $y$ with $y > x$. The filtration of $K_{x,V}$ whose terms are the subfunctors generated by the values at objects which are initial sequences in a list of objects $y_1, y_2, \ldots$ where $y_i > y_j$ implies $i < j$ has the property that each factor is supported on a single isomorphism class. It will be a direct sum of $\Delta_{y,W}$ (and hence have a $\Delta$-filtration) if and only if the evaluation on each object is projective. This is equivalent to requiring that each $K_{x,V}(y)$ be projective, and so if this condition is satisfied $K_{x,V}$ has a $\Delta$-filtration as required. Conversely, if $K_{x,V}$ has a $\Delta$-filtration we see that for each $y$, $K_{x,V}$ has a filtration with factors $\Delta_{y,W}(y) = P_W$. Since these modules are projective, it is equivalent to require that $K_{x,V}(y)$ be projective.

We conclude from this that $k\mathcal{C}$ is standardly stratified if and only if each evaluation $P_{x,V}(y)$ is a projective $k\,\mathrm{Aut}(y)$-module. By Proposition 2.3 each functor $k[\mathrm{Hom}(x,-)]$ is a direct sum of functors $P_{x,V}$, each occuring with non-zero multiplicity in the sum, and so every evaluation $P_{x,V}(y)$ is projective if and only if every $k[\mathrm{Hom}(x,y)]$ is a projective $k\,\mathrm{Aut}(y)$-module. Now $k[\mathrm{Hom}(x,y)]$ is a permutation module and it is projective if and only if every stabilizer in the action of $\mathrm{Aut}(y)$ on $\mathrm{Hom}(x,y)$ has order invertible in $k$ (because in characteristic $p$, for example, it is projective if and only if it is projective on restriction to a Sylow $p$-subgroup, where it is free; see [4]). This completes the proof. $\square$

**Corollary 2.6.** *Let $\mathcal{C}$ be a finite EI category in which every morphism is an epimorphism and let $k$ be a field. Then $k\mathcal{C}$ is standardly stratified with respect to the canonical preorder on $\Lambda$.*

*Proof.* The stabilizer condition which appears in the statement of Theorem 2.5 is satisfied whenever $\mathcal{C}$ has the property that all morphisms are epimorphisms. In such a situation, if $\alpha : x \to y$ is a morphism in $\mathcal{C}$ and $\theta_1, \theta_2 \in \mathrm{Aut}(y)$ then $\theta_1\alpha = \theta_2\alpha$ implies $\theta_1 = \theta_2$, so that $\mathrm{Stab}_{\mathrm{Aut}(y)}(\alpha) = \{1\}$, or in other words $\mathrm{Aut}(y)$ acts freely on $\mathrm{Hom}(x,y)$. $\square$

Various categories constructed from the subgroups of a group satisfy the condition that all morphisms are epimorphism, and we mention some of these now. Let $\mathcal{S}$ be a set of subgroups of a group $G$. We define the *transporter category* $\mathcal{T}_{\mathcal{S}}$ to have as its objects the members of $\mathcal{S}$, and morphisms $\mathrm{Hom}(H,K) = N_G(H,K) = \{g \in G \bigm| {}^gH \subseteq K\}$. This latter set is the *transporter* of $H$ into $K$, and when $H = K$ it is simply the normalizer of $H$ in $G$ so that $\mathcal{T}_{\mathcal{S}}$ is an EI-category. The composition of morphisms is group multiplication. It is clear that all morphisms in $\mathcal{T}_{\mathcal{S}}$ are both epimorphisms and monomorphisms, since if $h_1g = h_2g$ or $gh_1 = gh_2$ then $h_1 = h_2$.

A related category is the *orbit category* $\mathcal{O}_{\mathcal{S}}$ associated to $\mathcal{S}$ in which the objects are the coset spaces $G/H$ where $H \in \mathcal{S}$ and the morphisms are the $G$-equivariant mappings. Because the image of each equivariant map $G/H \to G/K$ is a union of orbits, and $G/K$ consists of only one orbit, each morphism is an epimorphism. In fact, it is well-known that all equivariant mappings $G/H \to G/K$ have the form $\alpha_g : xH \mapsto xg^{-1}K$ where $g \in N_G(H,K)$. The assignment $g \mapsto \alpha_g$ specifies a surjective mapping $N_G(H,K) \to \mathrm{Hom}_{\mathcal{O}_{\mathcal{S}}}(G/H, G/K)$ in which

$$\{x \in N_G(H,K) \bigm| \alpha_x = \alpha_g\} = Kg,$$

so that $K\backslash N_G(H,K)$ is in bijection with $\mathrm{Hom}_{\mathcal{O}_{\mathcal{S}}}(G/H, G/K)$. We see from this that the orbit category is obtained from the transporter category by replacing $N_G(H,K)$ with $K\backslash N_G(H,K)$ as the morphism set from $H$ to $K$. Orbit categories appear widely in the study of group actions on spaces. They are at the heart of Bredon's notion of a coefficient system and in the construction of Bredon homology (see [6, 9, 24]). They are also important in the construction of approximations to classifying spaces of groups, and from the extensive literature on this topic we may select [15, 12, 7] to illustrate the application.

We also construct the *Frobenius category* or *Quillen category* $\mathcal{F}_{\mathcal{S}}$ associated to the set of subgroups $\mathcal{S}$ (see [25, Sect. 47]). This is a category which plays a crucial role in the notion of a $p$-local finite group [7], and in the case when $\mathcal{S}$ consists of the elementary abelian $p$-subgroups it was used by Quillen in his stratification theorem for group cohomology. The category $\mathcal{F}_{\mathcal{S}}$ has the elements of $\mathcal{S}$ as its objects, and $\mathrm{Hom}_{\mathcal{F}_{\mathcal{S}}} = N_G(H,K)/C_G(H)$. The morphisms may be identified with the set of group homomorphisms $H \to K$ which are of the form 'conjugation by $g$' for some $g \in G$. Such group homomorphisms are monomorphisms, and so all morphisms in $\mathcal{F}_{\mathcal{S}}$ are monomorphisms. Thus all morphisms in the opposite category $\mathcal{F}_{\mathcal{S}}^{\mathrm{op}}$ are epimorphisms.

In view of this we obtain the following corollary.

**Corollary 2.7.** *Over any field, representations of the orbit category $\mathcal{O}_{\mathcal{S}}$, the transporter category $\mathcal{T}_{\mathcal{S}}$, its opposite $\mathcal{T}_{\mathcal{S}}^{\mathrm{op}}$ and the opposite $\mathcal{F}_{\mathcal{S}}^{\mathrm{op}}$ of the Frobenius category are all standardly stratified with respect to the natural preorder on the simple modules specified by $S_{x,V} \leq S_{y,W}$ if and only if there exists a morphism $x \to y$.*

## 3. Dlab's theorem and the Ringel dual

We suppose that $A$ is a finite-dimensional algebra over a field $k$, whose simple modules are parametrized by a set $\Lambda$. The theory of standardly-stratified algebras described in (for example) [1, 2, 10, 16, 17, 30] was developed under the hypothesis that $\Lambda$ is a poset. In fact the key assertions of this development hold when $\Lambda$ is a preordered set, and we wish to indicate in this section that this is so. A development in this generality has also been given in [13] which appeared as a preprint a few months earlier than the present account and was unknown to me until after this paper was written. In [13] a different approach is taken to proving Dlab's theorem.

We start by illustrating the kind of possibility that is allowed if $\Lambda$ is a preordered set, but not a poset. Consider an algebra $A$ with two simple modules labelled 1 and 2 whose regular representation is $\begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\2\end{smallmatrix}$ (for example, the group algebra $\mathbb{F}_3 S_3$). We see that $A$ is not standardly stratified with respect to any partial order on $\Lambda = \{1, 2\}$. For if $1 < 2$ then $\Delta_1 = 1$, $\Delta_2 = \begin{smallmatrix}2\\1\end{smallmatrix}$ and this does not give a stratification. Neither does the possibility $2 < 1$, which is similar. Equally, $A$ is not standardly stratified if $\Lambda$ has the partial order in which 1 and 2 are not comparable. On the other hand we may use the trivial preorder in which $1 \leq 2$ and $2 \leq 1$. Now $\Delta_1 = \begin{smallmatrix}1\\2\\1\end{smallmatrix}$, $\Delta_2 = \begin{smallmatrix}2\\1\\2\end{smallmatrix}$ and $A$ is standardly stratified.

The trivial stratification just considered gives no extra information about an algebra, since every algebra has such a stratification, and it might be thought merely a matter of terminology whether one says an algebra is trivially stratified, or not stratified at all. However this trivial example underlies non-trivial stratifications of other algebras which are useful to us. For example an algebra with regular representation

$$A = \begin{array}{ccc} & 1 & \\ 2 & & 3 \\ 1 & & \end{array} \oplus \begin{array}{ccc} & 2 & \\ 1 & & 4 \\ 2 & & \end{array}$$

is standardly stratified if $1 \leq 2 \leq 1 < 3 \leq 4 \leq 3$, but not standardly stratified with any partial order on $\Lambda = \{1, 2, 3, 4\}$.

We now summarize the theory of the Ringel dual and Dlab's theorem in the situation where $\Lambda$ is a preordered set. The main task is to make definitions of the standard and costandard objects correctly as well as to impose a certain condition on $\Lambda$. After that, the arguments are just the same as the well-known ones which already appear in the literature. In presenting them here we claim no originality, except in so far as to observe that in some cases these arguments hold in greater generality than their original context. It seems valuable to have a statement of these results in the generality that $\Lambda$ is a preordered set, since otherwise the question remains as to whether they do indeed hold.

Let $A$ be a finite-dimensional algebra over a field $k$, whose simple modules are parametrized as $S_\lambda$ where $\lambda \in \Lambda$, a preordered set. As before, if $\lambda, \mu \in \Lambda$ we will write $\lambda < \mu$ to mean $\lambda \leq \mu$ and $\mu \not\leq \lambda$. We write $\lambda \sim \mu$ if $\lambda \leq \mu$ and $\mu \leq \lambda$. Let $P_\lambda$ and $I_\lambda$ be the projective cover and injective hull of $S_\lambda$. We define modules $\Delta_\lambda$, $\overline{\Delta}_\lambda$, $\nabla_\lambda$ and $\overline{\nabla}_\lambda$ in this context as follows. We have already defined the *standard module* $\Delta_\lambda$ to be the largest quotient of $P_\lambda$ all of whose composition factors are isomorphic to $S_\mu$ with $\mu \leq \lambda$. We define the *proper standard module* $\overline{\Delta}_\lambda$ to be the largest quotient of $P_\lambda$ whose composition factors $S_\mu$ all satisfy $\mu < \lambda$, except for a single $S_\lambda$. Similarly the *costandard module* $\nabla_\lambda$ is the largest submodule of $I_\lambda$ with composition factors $S_\mu$ where $\mu \leq \lambda$ and the *proper costandard module* $\overline{\nabla}_\lambda$ is the largest submodule of $I_\lambda$ with composition factors $S_\mu$ where $\mu < \lambda$ except for a single $S_\lambda$. We observe that if $\lambda$ is minimal in $\Lambda$ then $\overline{\Delta}_\lambda = \overline{\nabla}_\lambda = S_\lambda$ is simple, and that if $\lambda$ is maximal in $\Lambda$ then $\Delta_\lambda = P_\lambda$ is projective and $\nabla_\lambda = I_\lambda$ is injective.

We denote by $\mathcal{F}(\Delta)$ the full subcategory of $A$-modules whose objects have a finite filtration in which each factor has the form $\Delta_\lambda$ for some $\lambda \in \Lambda$, and similarly we define $\mathcal{F}(\overline{\Delta})$, $\mathcal{F}(\nabla)$ and $\mathcal{F}(\overline{\nabla})$. We see that if $A$ is standardly-stratified in the sense of the definition in the last section then $A \in \mathcal{F}(\Delta)$, but it is not apparent that the converse need be true in this generality. The problem is that it need not be the case that Ext groups between the $\Delta$s vanish appropriately. As an example consider (again) the algebra whose regular representation is ${}_AA = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix}$ with two simple modules: 1 and 2. Taking this time the preorder in which 1 and 2 are not comparable, we see that $\Delta_1 = 1$ and $\Delta_2 = 2$ and that $A \in \mathcal{F}(\Delta)$, but that $A$ is not standardly stratified with this preorder.

To remedy this, we use the fact that there is associated to $\Lambda$ a poset $[\Lambda]$ whose elements are the equivalence classes $[\lambda]$ of elements of $\lambda$ under $\sim$ and where $[\lambda] \leq [\mu]$ if and only if $\lambda \leq \mu$. We may place the elements of $[\Lambda]$ in some order $[\Lambda] = \{\rho_1, \rho_2, \ldots, \rho_n\}$ so that $\rho_i \leq \rho_j$ implies $i \leq j$ and use this to define a new preorder on $\Lambda$: $\lambda \leq' \mu$ if and only if $\lambda \in \rho_i$, $\mu \in \rho_j$ with $i \leq j$. If $A$ is standardly stratified with respect to $(\Lambda, \leq)$ then it is also standardly-stratified with respect to $(\Lambda, \leq')$ with the same standard and proper standard modules (as we may easily see by an elementary argument), and so for many purposes we may work with $(\Lambda, \leq')$ instead of $(\Lambda, \leq)$. A number of the results in this section will only be true under the hypothesis that $\leq=\leq'$, which is equivalent to saying that the poset associated to $(\Lambda, \leq)$ is a linear order, or that in every pair of elements of $\Lambda$ the elements are comparable. This idea is present in [8, 2.2.3].

The standard and proper standard modules may also be defined in terms of idempotents. Frequently this is done elsewhere in the literature under the assumption that $\Lambda$ is a poset, but it may also be done under the weaker assumption that $\Lambda$ is a preordered set. To make the definitions work we suppose that the poset $[\Lambda] = \{\rho_1, \rho_2, \ldots, \rho_n\}$ associated to $\Lambda$ is linearly ordered. For each $\lambda \in \Lambda$ let $e_\lambda \in A$ be a primitive idempotent corresponding to the simple module $S_\lambda$. For each equivalence class $\rho_i$ of elements of $\Lambda$ we define an idempotent $E_i = \sum_{\lambda\in\rho_i} e_\lambda$, so that $E_i$ is the sum of as many primitive idempotents as there are elements in the equivalence class $\rho_i$. Let us put $\epsilon_i = \sum_{i\leq j} E_j$. Thus for $\lambda \in \rho_i$ we have

$$\Delta_\lambda \cong (Ae_\lambda)/(A\epsilon_{i+1}(\operatorname{Rad} A)e_\lambda)$$

and

$$\overline{\Delta}_\lambda \cong (Ae_\lambda)/(A\epsilon_i(\operatorname{Rad} A)e_\lambda).$$

The way to see this is to observe that $Ae_\lambda \cong P_\lambda$, the projective cover of $S_\lambda$, so that $(\operatorname{Rad} A)e_\lambda$ is the unique maximal submodule of $P_\lambda$ and $A\epsilon_{i+1}(\operatorname{Rad} A)e_\lambda$ is the trace in $(\operatorname{Rad} A)e_\lambda$ of the projective modules $P_\mu$ where $\lambda < \mu$. Similarly $A\epsilon_i(\operatorname{Rad} A)e_\lambda$ is the trace in $(\operatorname{Rad} A)e_\lambda$ of the projective modules $P_\mu$ where $\lambda \leq \mu$.

We list some basic properties of these modules.

**Proposition 3.1.** *Let $A$ be a finite-dimensional $k$-algebra whose simple modules are parametrized (up to isomorphism) by a preordered set $\Lambda$ for which the associated poset $[\Lambda]$ is linearly ordered. The modules we have defined satisfy the following properties.*

1. $\operatorname{Hom}_A(\Delta_\lambda, \nabla_\mu) \neq 0$ *implies* $\lambda \sim \mu$.
2. $\operatorname{Hom}_A(\Delta_\lambda, \overline{\nabla}_\mu) \neq 0$ *implies* $\lambda = \mu$, *and also* $\operatorname{Hom}_A(\overline{\Delta}_\mu, \nabla_\lambda) \neq 0$ *implies* $\lambda = \mu$.
3. $\operatorname{Hom}_A(\overline{\Delta}_\lambda, \overline{\nabla}_\mu) \neq 0$ *implies* $\lambda = \mu$.
4. *If* $\mu \leq \lambda$ *then* $\operatorname{Ext}^1_A(\Delta_\lambda, S_\mu) = 0$ *and* $\operatorname{Ext}^1_A(S_\mu, \nabla_\lambda) = 0$.
5. $\operatorname{Ext}^1_A(\Delta_\lambda, \Delta_\mu) \neq 0$ *implies* $\lambda < \mu$, *and also* $\operatorname{Ext}^1_A(\nabla_\mu, \nabla_\lambda) \neq 0$ *implies* $\lambda < \mu$.
6. $\operatorname{Ext}^1_A(\Delta_\lambda, \overline{\Delta}_\mu) \neq 0$ *implies* $\lambda < \mu$, *and also* $\operatorname{Ext}^1_A(\overline{\nabla}_\mu, \nabla_\lambda) \neq 0$ *implies* $\lambda < \mu$.

(7) $\mathrm{Ext}^1_A(\overline{\Delta}_\lambda, \Delta_\mu) \neq 0$ *implies* $\lambda \leq \mu$, *and also* $\mathrm{Ext}^1_A(\nabla_\mu, \overline{\nabla}_\lambda) \neq 0$ *implies* $\lambda \leq \mu$.

(8) $\mathrm{Ext}^1_A(\overline{\Delta}_\lambda, \overline{\Delta}_\mu) \neq 0$ *implies* $\lambda \leq \mu$, *and also* $\mathrm{Ext}^1_A(\overline{\nabla}_\mu, \overline{\nabla}_\lambda) \neq 0$ *implies* $\lambda \leq \mu$.

(9) $\mathrm{Ext}^1_A(\Delta_\lambda, \nabla_\mu) = 0$ *for all* $\lambda, \mu \in \Lambda$.

(10) $\mathrm{Ext}^1_A(\Delta_\lambda, \overline{\nabla}_\mu) = 0$ *and* $\mathrm{Ext}^1_A(\overline{\Delta}_\lambda, \nabla_\mu) = 0$ *for all* $\lambda, \mu \in \Lambda$.

(11) $\mathrm{Ext}^1_A(\overline{\Delta}_\lambda, \overline{\nabla}_\mu) \neq 0$ *implies* $\lambda \sim \mu$.

*Proof.* In each case where there are two statements they are equivalent to each other by the duality between modules for $A$ and its opposite algebra. We only prove the first statement. All of the arguments are routine, being similar to the ones which prove 1.2 and 1.3 of [11] or 2.2.2 and 2.2.8 of [8], for example.

(1) If there is a non-zero homomorphism $\phi : \Delta_\lambda \to \nabla_\mu$ we deduce that $S_\lambda$ is a composition factor of $\nabla_\mu$ and $S_\mu$ is a composition factor of $\Delta_\lambda$. Thus $\lambda \leq \mu$ and $\mu \leq \lambda$ so $\lambda \sim \mu$.

(2) If there is a non-zero homomorphism $\phi : \Delta_\lambda \to \overline{\nabla}_\mu$ then $\mu \leq \lambda$ as in (1), but now if $\lambda \neq \mu$ then $\lambda < \mu$ since $S_\lambda$ is a composition factor of $\overline{\nabla}_\mu$. This would imply $\lambda < \mu \leq \lambda$, which is not possible.

(3) If there is a non-zero homomorphism $\phi : \overline{\Delta}_\lambda \to \overline{\nabla}_\mu$ then either $\lambda = \mu$ or else $\lambda < \mu$ and $\mu < \lambda$, which is not possible.

(4) These follow from the fact that $\Delta_\lambda$ is the largest quotient of $P_\lambda$ with composition factors $\leq \lambda$, and that $\nabla_\lambda$ is the largest submodule of $I_\lambda$ with composition factors $\leq \lambda$. It may also be proved by the argument of (5).

(5) We use the short exact sequence $0 \to K_\lambda \to P_\lambda \to \Delta_\lambda \to 0$ to compute Ext. The semisimple top of $K_\lambda$ only has composition factors $S_\nu$ with $\lambda < \nu$ so $\mathrm{Hom}_A(K_\lambda, \Delta_\mu) \neq 0$ implies $\Delta_\mu$ has such a composition factor, and $\lambda < \nu \leq \mu$.

(6) This is similar to (5).

(7) We use the short exact sequence $0 \to \overline{K}_\lambda \to P_\lambda \to \overline{\Delta}_\lambda \to 0$ to compute Ext, where the composition factors of the semisimple top of $\overline{K}_\lambda$ are $S_\nu$ with $\lambda \leq \nu$. Thus $\mathrm{Hom}_A(\overline{K}_\lambda, \Delta_\mu) \neq 0$ implies $\Delta_\mu$ has such a composition factor, and $\lambda \leq \nu \leq \mu$.

(8) This is similar to (7).

(9) The argument for this is given in 2.2.8 of [8].

(10) This and (9) are perhaps the most interesting arguments. We use the short exact sequence $0 \to K_\lambda \to P_\lambda \to \Delta_\lambda \to 0$ as before to compute Ext, and because $\mathrm{Hom}_A(K_\lambda, \overline{\nabla}_\mu) \neq 0$ implies $\lambda < \mu$ we deduce that if $\mathrm{Ext}^1_A(\Delta_\lambda, \overline{\nabla}_\mu) \neq 0$ then $\lambda < \mu$.

We may also use the injective presentation $0 \to \overline{\nabla}_\mu \to I_\mu \to \overline{C}_\mu \to 0$, where the socle of $\overline{C}_\mu$ has composition factors $S_\nu$ with $\mu \leq \nu$, so that

$\mathrm{Hom}_A(\Delta_\lambda, \overline{C}_\mu) \neq 0$ implies $\mu \le \nu \le \lambda$ so $\mu \le \lambda$. Because we cannot simultaneously have $\lambda < \mu$ and $\mu \le \lambda$ we deduce that $\mathrm{Ext}^1_A(\Delta_\lambda, \overline{\nabla}_\mu) = 0$.

(11) Here $\mathrm{Hom}_A(\overline{K}_\lambda, \overline{\nabla}_\mu) \neq 0$ implies $\lambda \le \mu$ and $\mathrm{Hom}_A(\overline{\Delta}_\lambda, \overline{C}_\mu) \neq 0$ implies $\mu \le \lambda$. □

**Corollary 3.2.** *Let $A$ be a finite-dimensional $k$-algebra whose simple modules are parametrized (up to isomorphism) by a preordered set $\Lambda$ for which the associated poset $[\Lambda] = \{\rho_1, \rho_2, \ldots, \rho_n\}$ is linearly ordered, and let the idempotents $\epsilon_i$ be defined as above.*

1. *Let $M$ be a finite-dimensional $A$-module. Then $M$ lies in $\mathcal{F}(\Delta)$ if and only if for every $i$ the module $(A\epsilon_i M)/(A\epsilon_{i+1} M)$ is a direct sum of modules $\Delta_\lambda$ with $\lambda \in \rho_i$.*
2. *$\mathcal{F}(\Delta)$ is closed under taking direct summands.*
3. *$A$ is standardly-stratified if and only if $A \in \mathcal{F}(\Delta)$, or equivalently, every finite-dimensional projective $A$ module lies in $\mathcal{F}(\Delta)$.*

*Proof.* (1) The argument is implicit in [8] Section 2.2. Assuming $M \in \mathcal{F}(\Delta)$, by (5) of Proposition 3.1 $M$ has a filtration in which the factors occur in an order compatible with the linear order on $[\Lambda]$. Such a filtration is a refinement of the trace filtration with terms $A\epsilon_i M$ and each factor is a direct sum of modules $\Delta_\lambda$ again by (5) of Proposition 3.1. The converse implication is easy.

(2) This is the usual argument, as in [23] Section 3 or [11] 1.4.

(3) From the definition we have that if $A$ is standardly stratified then $A \in \mathcal{F}(\Delta)$. Conversely if $A \in \mathcal{F}(\Delta)$ then each indecomposable projective $P$ lies in $\mathcal{F}(\Delta)$ and by part (1) the filtration with terms $A\epsilon_i P$ has $\Delta$ factors. Since $P$ is indecomposable projective the top factor must be $\Delta_\lambda$ for some $\lambda$, and the rest of the factors must be $\Delta_\mu$ with $\mu > \lambda$ by the property of this filtration. □

We also have the following:

**Corollary 3.3.** *Let $A$ be a finite-dimensional $k$-algebra whose simple modules are parametrized by a preordered set $\Lambda$ for which the associated poset $[\Lambda]$ is linearly ordered.*

1. *$\mathcal{F}(\Delta)$ and $\mathcal{F}(\overline{\Delta})$ are closed under kernels of epimorphisms.*
2. *$\mathcal{F}(\nabla)$ and $\mathcal{F}(\overline{\nabla})$ are closed under cokernels of monomorphisms.*

*Proof.* Parts (1) and (2) are dual to each other and only one of them needs to be proved. The statement about $\mathcal{F}(\Delta)$ follows as in 1.5 of [11] and the statement about $\mathcal{F}(\overline{\Delta})$ follows as in 3.2 of [1]. □

Dlab showed [10] under the hypothesis that $\Lambda$ itself is linearly-ordered that the left regular representation ${}_AA$ lies in $\mathcal{F}(\Delta)$ if and only if the right regular representation $A_A$ lies in $\mathcal{F}(\overline{\Delta})$ (where now the $\overline{\Delta}$s are right modules), or what is the same on considering ordinary duality with respect to the field $k$, that all finitely-generated injective left $A$-modules lie in $\mathcal{F}(\overline{\nabla})$. A similar thing works when $\Lambda$ is a preordered set. We state a result which is a combination of the statements which appear in [1, Theorem 3.1] and [17, Theorem 3]. Rather than work with left $\Delta$s and right $\overline{\Delta}$s (as is done elsewhere) we choose to work entirely with left modules, so that instead of the right modules $\overline{\Delta}$ we work with their vector space duals, which are the left modules $\overline{\nabla}$.

**Theorem 3.4.** *(Dlab's Theorem) Let $A$ be a finite-dimensional $k$-algebra whose simple modules are parametrized (up to isomorphism) by a preordered set $\Lambda$ for which the associated poset $[\Lambda]$ is linearly ordered. The following statements are all equivalent to the statement that $A$ is standardly stratified.*

($\Delta$1) $\mathcal{F}(\Delta)$ *contains all finitely-generated projective modules.*
($\overline{\nabla}$1) $\mathcal{F}(\overline{\nabla})$ *contains all finitely-generated injective modules.*
(2) $\mathrm{Ext}^2_A(\Delta_\lambda, \overline{\nabla}_\mu) = 0$ *for all* $\lambda, \mu \in \Lambda$.
($\Delta$3) $\mathcal{F}(\Delta) = \{X \mid \mathrm{Ext}^1_A(X, \overline{\nabla}_\mu) = 0\}$ *for all* $\mu \in \Lambda$.
($\overline{\nabla}$3) $\mathcal{F}(\overline{\nabla}) = \{X \mid \mathrm{Ext}^1_A(\Delta_\mu, X) = 0\}$ *for all* $\mu \in \Lambda$.

*These conditions are in turn equivalent to the following seemingly stronger conditions:*

(2′) $\mathrm{Ext}^k_A(\Delta_\lambda, \overline{\nabla}_\mu) = 0$ *for all* $\lambda, \mu \in \Lambda$ *and all* $k \geq 1$.
($\Delta$3′) $\mathcal{F}(\Delta) = \{X \mid \mathrm{Ext}^k_A(X, \overline{\nabla}_\mu) = 0$ *for all* $\mu \in \Lambda$ *and all* $k \geq 1\}$.
($\overline{\nabla}$3′) $\mathcal{F}(\overline{\nabla}) = \{X \mid \mathrm{Ext}^k_A(\Delta_\lambda, X) = 0$ *for all* $\lambda \in \Lambda$ *and all* $k \geq 1\}$.

*Proof.* We have observed in part (3) of Corollary 3.2 that condition ($\Delta$1) is equivalent to the condition that $A$ is standardly stratified. The proofs of the equivalences which appear in [1] and [17] go through verbatim with the modules $\Delta$ and $\overline{\nabla}$ which we have defined here, since these modules satisfy the same formal properties as the ones considered in [1] and [17].

Certain implications are immediate, as in [1] and [17]. These are ($\Delta$3′) $\Rightarrow$ (2′) $\Rightarrow$ (2), ($\overline{\nabla}$3′) $\Rightarrow$ (2′), ($\Delta$3) $\Rightarrow$ ($\Delta$1) and ($\overline{\nabla}$3) $\Rightarrow$ ($\overline{\nabla}$1).

We prove ($\Delta$1) $\Rightarrow$ (2′) by an inductive argument involving dimension-shifting, as is done at the bottom of page 101 of [17], using part 10 of Proposition 3.1 and part 1 of Corollary 3.3; ($\overline{\nabla}$1) $\Rightarrow$ (2′) is proved similarly.

($\Delta$3) $\Rightarrow$ ($\Delta$3′) and ($\overline{\nabla}$3) $\Rightarrow$ ($\overline{\nabla}$3′) are also proved in this fashion, as is indicated on page 102 of [17].

The harder parts of the proof are (2) $\Rightarrow$ ($\Delta$3) which proceeds as on pages 102 and 103 of [17], and (2) $\Rightarrow$ ($\overline{\nabla}$3), which proceeds as on pages 5 and 6 of [1]. $\square$

By analogy with [2, Theorem 1.6] we now have:

**Theorem 3.5.** *Let $A$ be a standardly-stratified algebra with respect to a preordered set $\Lambda$ for which the associated poset $[\Lambda]$ is linearly ordered. Then*

(1) *$\mathcal{F}(\Delta)$ is a functorially finite and resolving subcategory of $A$-modules.*
(2) *$\mathcal{F}(\overline{\nabla})$ is a covariantly finite and coresolving subcategory of $A$-modules.*

*Proof.* We see from Corollary 3.3 and conditions ($\Delta$1) and ($\overline{\nabla}$1) of Dlab's Theorem that $\mathcal{F}(\Delta)$ is resolving and $\mathcal{F}(\overline{\nabla})$ is coresolving. The fact that $\mathcal{F}(\Delta)$ is functorially finite follows from [23] using part (5) of Proposition 3.1. The fact that $\mathcal{F}(\overline{\nabla})$ is covariantly finite follows from (1), condition ($\overline{\nabla}$3) of Dlab's theorem and [5, 3.3]. $\square$

The Ringel dual of a standardly-stratified algebra was constructed in [2] (under the hypothesis that $\Lambda$ is a poset) and it was also constructed independently in [29, Theorem 9.1] (without this hypothesis), where as part of a more general construction it was shown that the Ringel dual of a standardly-stratified algebra is again standardly-stratifed. The approach used in [29] was to classify the indecomposable Ext-injective objects in $\mathcal{F}(\Delta)$ as the 'Ext-injective hulls' $T_\lambda$ of the $\Delta_\lambda$, without using the property that they are precisely the indecomposable objects in $\mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\nabla})$.

In [29, Theorem 9.1] we gave a sufficient condition for the Ringel dual algebra to be standardly stratified, but unfortunately there was a significant typographical error in the statement of the theorem, and we take the opportunity to point this out. In that theorem the modules which are here called $\Delta_\lambda$ were called $\Theta(\lambda)$ and the hypothesis $\mathrm{Hom}(\Theta(\lambda), \Theta(\rho)) = 0$ unless $\lambda > \rho$ was made. It should have read '$\mathrm{Hom}(\Theta(\lambda), \Theta(\rho)) = 0$ unless $\lambda \geq \rho$', and the proof used this latter condition. We comment also that in [29, Theorem 9.10] the $\Theta(\lambda)$ were indexed by the opposite preorder to the one used here.

In fact, the usual development of the properties of the Ringel dual, as described for example in [2], holds in the situation where $\Lambda$ is a preordered set, and we now summarize it. The Ringel dual is the algebra $B = \mathrm{End}_A(T)$ where $T = \bigoplus_{\lambda \in \Lambda} T_\lambda$. We adopt the convention consistent with [29], but opposite to a frequently used convention, that homomorphisms are applied and composed on the left. Thus starting with a left $A$-module $T$, we regard $T$ also as a *left* $B$-module. There are four contravariant functors we consider, namely $F_A = \mathrm{Hom}_A(-, T) : A\text{-mod} \to B\text{-mod}$, $G_A = D\,\mathrm{Hom}_A(T, -) : A\text{-mod} \to B\text{-mod}$ where $D(X) = \mathrm{Hom}_k(X, k)$, and also $F_B = \mathrm{Hom}_B(-, T)$, $G_B = D\,\mathrm{Hom}_B(T, -)$ which go in the opposite direction to $F_A$ and $G_A$. At this point we introduce the notation of placing a superscript $A$ on a symbol that denotes an $A$-module, and a superscript $B$ on a symbol that denotes a $B$-module. The exception to this is $T$, which is both an $A$-module and a $B$-module, and appears without a superscript.

**Theorem 3.6.** *Let $A$ be a finite-dimensional $k$-algebra whose simple modules are parametrized (up to isomorphism) by a preordered set $\Lambda$ for which the associated poset $[\Lambda]$ is linearly ordered. The isomorphism types of indecomposable objects of $\mathcal{F}(\Delta^A) \cap \mathcal{F}(\overline{\nabla}^A)$ are parametrized as $T_\lambda$ where $\lambda \in \Lambda$. For each $\lambda$ there are short exact sequences*

$$0 \to \Delta^A_\lambda \to T_\lambda \to X^A_\lambda \to 0$$

$$0 \to Y^A_\lambda \to T_\lambda \to \overline{\nabla}^A_\lambda \to 0$$

*where $X^A_\lambda$ has a filtration with factors $\Delta^A_\mu$ where $\mu < \lambda$ and $Y^A_\lambda$ has a filtration with factors $\overline{\nabla}^A_\mu$ where $\mu \le \lambda$. (In fact $T^A_\lambda$ is the Ext-injective hull in $\mathcal{F}(\Delta^A)$ of $\Delta^A_\lambda$ (in the sense of* [29]*).) The module $T = \bigoplus_{\lambda\in\Lambda} T^A_\lambda$ is a tilting module for $A$. The dual algebra $B = \mathrm{End}_A(T)$ is again standardly stratified, with preordered set $\Lambda^{\mathrm{op}}$, standard modules $\Delta^B_\lambda = F_A(\Delta^A_\lambda)$ and proper costandard modules $\overline{\nabla}^B_\lambda = G_A(\overline{\nabla}^A_\lambda)$. Up to isomorphism the indecomposable summands of $T$ as a $B$-module are precisely the indecomposable modules in $\mathcal{F}(\Delta^B) \cap \mathcal{F}(\overline{\nabla}^B)$, $T$ is a tilting module for $B$, and $A \cong \mathrm{End}_B(T)$ which is Morita equivalent to the dual algebra of $B$. The functors $F_A$ and $F_B$ provide a duality between $\mathcal{F}(\Delta^A)$ and $\mathcal{F}(\Delta^B)$ which interchanges $P^A_\lambda$ with $T^B_\lambda$ and $T^A_\lambda$ with $P^B_\lambda$. The functors $G_A$ and $G_B$ provide a duality between $\mathcal{F}(\overline{\nabla}^A)$ and $\mathcal{F}(\overline{\nabla}^B)$ which interchanges $I^A_\lambda$ with $T^B_\lambda$ and $T^A_\lambda$ with $I^B_\lambda$.*

*Proof.* We will describe a way to approach this which at many points is similar to the line of development in [2]. For a number of arguments we choose to quote from [29], but other references are possible.

The modules in $\mathcal{F}(\Delta^A) \cap \mathcal{F}(\overline{\nabla}^A)$ are the Ext-injective objects of $\mathcal{F}(\Delta^A)$ by Dlab's Theorem, and these were classified in [29, 8.6] as the Ext-injective hulls of the $\Delta^A_\lambda$. This provides the first of the short exact sequences in the statement of the theorem and it follows from the construction there that $X^A_\lambda$ has a filtration with factors $\Delta^A_\mu$ where $\mu < \lambda$. It follows from [29, Theorem 9.1] that $B$ is standardly-stratified with preordered set $\Lambda^{\mathrm{op}}$ (according to the convention we are using here) and with standard modules $\Delta_\lambda^B = F(\Delta_\lambda^A)$.

We next show that $T$ is a tilting module, for which we must show that $T$ has finite projective dimension, $\mathrm{Ext}_A^i(T,T) = 0$ for all $i > 0$, and that the regular representation ${}_AA$ has a finite resolution by modules which are summands of direct sums of $T$. In fact every object of $\mathcal{F}(\Delta^A)$ has both a finite resolution by projective modules and also a finite resolution by Ext-injective modules, from the properties of Ext-projective and Ext-injective hulls developed in [29] (for example), and the Ext-injective hulls are always summands of a direct sum of copies of $T$. The fact that $\mathrm{Ext}_A^i(T,T) = 0$ for all $i > 0$ is immediate from $(\Delta 3')$ or $(\overline{\nabla}3')$ of Dlab's theorem.

Because $T$ is a tilting module the canonical map $A \to \mathrm{End}_B(T)$ is an isomorphism, and so the isomorphism types of projective $A$-modules biject with the isomorphism types of summands of $T$ as a $B$-module, and these are parametrized by $\Lambda$. Since the indecomposable objects of $\mathcal{F}(\Delta^B)\cap\mathcal{F}(\overline{\nabla}^B)$ are also parametrized by $\Lambda$, we will know that, as a $B$-module, $T$ is the sum of all of them (perhaps taken with multiplicity) provided we can show that $T$ is Ext-injective in $\mathcal{F}(\Delta^B)$. For this it suffices to show that $\mathrm{Ext}_B^1(\Delta_\lambda^B, T) = 0$ for all $\lambda \in \Lambda$. Take an Ext-injective resolution

$$0 \to \Delta_\lambda^A \to T_0^A \to \cdots \to T_n^A \to 0$$

where the $A$-modules $T_i^A$ are direct sums of summands of $T$. Applying the functor $F_A$, which is exact on $\mathcal{F}(\Delta^A)$ and sends summands of $T$ to projectives, we obtain a projective resolution

$$0 \leftarrow \Delta_\lambda^B \leftarrow P_0^B \leftarrow \cdots \leftarrow P_n^B \leftarrow 0$$

and compute $\mathrm{Ext}_B^*(\Delta_\lambda^B, T)$ as the homology $H_*(\mathrm{Hom}_B(P_\bullet^B, T))$. Since $\mathrm{Hom}_B(-,T) = F_B$ is an equivalence between the projective $B$-modules on the one hand and the sums of summands of $T$ on the other, we see that $\mathrm{Hom}_B(P_\bullet^B, T)$ is the original resolution of $\Delta_\lambda^A$, which is acyclic above degree 0. This shows that $\mathrm{Ext}_B^1(\Delta_\lambda^B, T) = 0$, and hence $A \cong \mathrm{End}_B(T)$ is indeed the dual algebra of $B$ (up to Morita equivalence). It follows also that $F_B(\Delta_\lambda^B) = \Delta_\lambda^A$.

We prove that $F_A$, $F_B$ are inverse dualities between $\mathcal{F}(\Delta^A)$ and $\mathcal{F}(\Delta^B)$. This is proven for example in [8, 3.8.2], but to save the reader the trouble of chasing references we present an argument. We show that both $F_A$ and $F_B$ are full and faithful. In the case of $F_A$, it is full and faithful on the full subcategory of $\mathcal{F}(\Delta^A)$ whose modules are the direct sums of summands of $T$, by the Fitting correspondence. Because every module in $\mathcal{F}(\Delta^A)$ has an Ext-injective resolution by such modules, $F_A$ is full and faithful on $\mathcal{F}(\Delta^A)$. The argument for $F_B$ is similar, and so $F_A$, $F_B$ are inverse dualities between $\mathcal{F}(\Delta^A)$ and $\mathcal{F}(\Delta^B)$ since they are exact and exchange $\Delta^A_\lambda$ with $\Delta^B_\lambda$.

It follows from tilting theory (see [21] Theorem 1.16 taken together with the duality isomorphism which appears on page 114 of [21], or [14]) that the functors $G_A$ and $G_B$ provide a duality between $\mathcal{F}(\overline{\nabla}^A)$ and $\mathcal{F}(\overline{\nabla}^B)$ and by the Fitting correspondence these functors interchange indecomposable summands of $T$ and indecomposable injective modules. More specifically, consider the indecomposable summand $T^A_\lambda$ of $T$ which has a $\Delta^A$-filtration with $\Delta^A_\lambda$ at the bottom, and let $e_\lambda \in B$ be the idempotent which is the composite of projection and inclusion $T \to T^A_\lambda \to T$. Then $F_A(T^A_\lambda) = \mathrm{Hom}_A(T^A_\lambda, T) \cong Be_\lambda$ and $\mathrm{Hom}_A(T, T^A_\lambda) \cong e_\lambda B$ are the indecomposable projective left and right $B$-modules corresponding to $\lambda$, from which we see that $G_A(T^A_\lambda) = I^B_\lambda$ is the injective left $B$-module parametrized by $\lambda$. Hence also $G_A(I^A_\lambda) = T^B_\lambda$.

Because the $\overline{\nabla}^A_\lambda$ and $\overline{\nabla}^B_\lambda$ are identified in $\mathcal{F}(\overline{\nabla}^A)$ and $\mathcal{F}(\overline{\nabla}^B)$ as the objects which do not have proper filtrations by any other objects in these categories, it follows that $G_A$ and $G_B$ interchanges them, so that $G_A(\overline{\nabla}^A_\lambda) = \overline{\nabla}^B_\rho$ for some $\rho$ depending on $\lambda$. We show that $\lambda = \rho$.

To do this, we show that in any sequence

$$0 \to Y^A \to T^A_\lambda \to \overline{\nabla}^A_\rho \to 0$$

where $Y \in \mathcal{F}(\overline{\nabla}^A)$ we must have $\lambda = \rho$. It will follow from this on applying the functor $G_A$ that $G_A(\overline{\nabla}^A_\lambda)$ is a submodule of $I^B_\lambda$, and hence must be $\overline{\nabla}^B_\lambda$. We start by observing that this same procedure shows that in any such sequence the right-hand term must be the same $\overline{\nabla}^A_\rho$, because its image under $G_A$ is always $\overline{\nabla}^B_\lambda$, being a submodule of $I^B_\lambda$. Thus all factors $\overline{\nabla}^A_\nu$ in a $\overline{\nabla}^A$-filtration of $Y^A$ must satisfy $\nu \le \rho$ by Proposition 3.1 part (8). Since $\Delta^A_\lambda$ is a submodule of $T^A_\lambda$, for some such $\nu$ we have $\mathrm{Hom}_A(\Delta^A_\lambda, \overline{\nabla}^A_\nu) \neq 0$ which implies $\lambda = \nu$ by Proposition 3.1 part (2), so $\lambda \le \rho$. At the same time, all factors $\Delta_\sigma$ of $T_\lambda$ in a $\Delta$-filtration have either $\sigma < \lambda$ or $\sigma = \lambda$ by Proposition 3.1 part (5) and there exists such a $\sigma$ for which $\mathrm{Hom}_A(\Delta_\sigma, \overline{\nabla}_\rho) \neq 0$. This implies that $\sigma = \rho$ by Proposition 3.1 part (2). Thus either $\rho < \lambda$, which is not possible by our previous conclusion, or $\rho = \lambda$, which is what we were aiming to prove. $\square$

We see that the $T^A_\lambda$ have a stronger property than the one stated in the last theorem, namely that if $\Delta^A_\rho \to T^A_\lambda$ is any monomorphism then necessarily $\rho = \lambda$ and the quotient has a $\Delta^A$-filtration, and also that if $T^A_\lambda \to \overline{\nabla}^A_\rho$ is any epimorphism then again $\lambda = \rho$ and the kernel has a $\overline{\nabla}^A$-filtration. This follows because on applying $F_A$ we obtain an epimorphism of $B$-modules $P^B_\lambda \to \Delta^B_\rho$ which forces $\lambda$ to equal $\rho$ and the kernel to have a $\Delta^B$-filtration, whereupon on applying $F_B$ we obtain that the cokernel of $\Delta^A_\lambda \to T^A_\lambda$ has a $\Delta^A$-filtration. The argument for $T^A_\lambda \to \overline{\nabla}^A_\rho$ is similar using $G_A$ and $G_B$.

There are further properties of these categories which we could mention and whose proofs we leave to the reader. Thus it follows from the fact that any module in $\mathcal{F}(\Delta^A)$ has a finite resolution by summands of direct sums of $T$, taken together with condition ($\overline{\nabla}$3) of Dlab's theorem, that $\mathcal{F}(\overline{\nabla}^A)$ may be characterized as $\{X \mid \mathrm{Ext}^1_A(T, X) = 0\}$. There is also a BGG reciprocity statement, which follows from (2) of Proposition 3.1.

## 4. Representations of the orbit category and Alperin's weight conjecture

In this section we focus on the orbit category $\mathcal{O}_\mathcal{S}$ of a finite group $G$ with respect to the set $\mathcal{S}$ of $p$-subgroups of $G$ where $p$ is the characteristic of the algebraically closed field $k$ and write $A = k\mathcal{O}_\mathcal{S}$. The orbit category has as its objects the transitive $G$-sets $G/H$ where $H \in \mathcal{S}$ and $\mathrm{Aut}_{\mathcal{O}_\mathcal{S}}(G/H) \cong N_G(H)/H$, so that the simple $k\mathcal{O}_\mathcal{S}$-modules are parametrized by pairs $(G/H, V)$ where $H \in \mathcal{S}$ is taken up to conjugacy and $V$ is a simple representation of $N_G(H)/H$ taken up to isomorphism. For brevity we will write such a pair as $(H, V)$, and write $S^A_{H,V}$, $P^A_{H,V}$, $\Delta^A_{H,V}$ for the corresponding simple, projective and standard $A$-modules, etc. When we consider these modules for the Ringel dual $B$ of $A$, they will acquire a superscript $B$.

In [3] Alperin defines a *weight* of $G$ to be a pair $(H, V)$ where $H$ is a $p$-subgroup of $G$ taken up to conjugacy and $V$ is a simple projective $k[N_G(H)/H]$-module, taken up to isomorphism. Alperin's weight conjecture states that the number of weights for $G$ at the prime $p$ equals the number of isomorphism types of irreducible $kG$-modules.

We see that the weights are a subset of the preordered set which parametrizes the simple representations of $\mathcal{O}_\mathcal{S}$. We will identify by categorical properties subsets of the $\mathcal{F}(\Delta)$ and $\mathcal{F}(\overline{\nabla})$ which biject with the simple $kG$-modules, and with the weights. In particular we identify by categorical properties the $\Delta^A_{H,V}$ when $H = 1$ and when $(H, V)$ is a weight. For completeness, we also identify the $\Delta^A_{H,V}$ when $H$ is a

Sylow $p$-subgroup of $G$. We conclude with reformulations of Alperin's conjecture.

In the following results the canonical modules to which we refer are the indecomposable modules $T^A_{H,V}$ in $\mathcal{F}(\Delta^A) \cap \mathcal{F}(\overline{\nabla}^A)$ and $T^B_{H,V}$ in $\mathcal{F}(\Delta^B) \cap \mathcal{F}(\overline{\nabla}^B)$.

**Theorem 4.1.** *Let $A = k\mathcal{O}_\mathcal{S}$ where $\mathcal{O}_\mathcal{S}$ is the orbit category of $G$ with respect to its $p$-subgroups, and let $B$ be the Ringel dual of $A$. The following are equivalent.*

(1) $\Delta^A_{H,V} = S^A_{H,V}$ *is a simple $k\mathcal{O}_\mathcal{S}$-module,*
(2) $\overline{\nabla}^A_{H,V} = I^A_{H,V}$ *is injective,*
(3) $\overline{\nabla}^B_{H,V} = T^B_{H,V}$ *is a canonical $B$-module.*
(4) $(H, V)$ *is a weight.*

*Proof.* To show that (1) $\Leftrightarrow$ (4) observe that $\Delta^A_{H,V}$ is the projective cover $P_V$ of $V$ as a module for $k[N_G(H)/H]$, concentrated at the object $G/H$ of $\mathcal{O}_\mathcal{S}$, and it is simple if and only if the projective cover is simple, which occurs if and only if $V$ is projective.

The equivalence (2) $\Leftrightarrow$ (3) is formal because of the duality between $\mathcal{F}(\overline{\nabla}^A)$ and $\mathcal{F}(\overline{\nabla}^B)$.

It remains to show (2) $\Leftrightarrow$ (4). We use the fact that $I^A_{H,V}$ is isomorphic to $\mathrm{Hom}_k(P^{A^{\mathrm{op}}}_{H,V}, k)$, so that $I^A_{H,V}(H) = P_V$. Assuming (2) we have

$$I^A_{H,V}(H) = P_V = \overline{\nabla}^A_{H,V}(V) = V,$$

so that $V$ is projective and $(H, V)$ is a weight. Conversely if $(H, V)$ is a weight then $I^A_{H,V}(H) = V = \overline{\nabla}^A_{H,V}(H)$. Apart from $S^A_{H,V}$, all the other composition factors $S^A_{K,W}$ of $I^A_{H,V}$ have $K <_G H$ and so $(K, W)$ is strictly smaller than $(H, V)$. We see from the definition of $\overline{\nabla}^A_{H,V}$ that $\overline{\nabla}^A_{H,V} = I^A_{H,V}$. □

In the proof of the next result we quote some of the theory of Mackey functors, as much as a convenience as for any other reason. As a guide to theory of Mackey functors we may refer to [28].

**Theorem 4.2.** *Let $A = k\mathcal{O}_\mathcal{S}$ where $\mathcal{O}_\mathcal{S}$ is the orbit category of $G$ with respect to its $p$-subgroups, and let $B$ be the Ringel dual of $A$. The following are equivalent.*

(1) $\Delta^A_{H,V} = T^A_{H,V}$ *is a canonical $k\mathcal{O}_\mathcal{S}$-module,*
(2) $\Delta^A_{H,V} = I^A_{H,V}$ *is an injective $A$-module,*
(3) $\Delta^B_{H,V} = P^B_{H,V}$ *is a projective $B$-module.*
(4) $H = 1$.

*Proof.* The equivalence (1) $\Leftrightarrow$ (3) is formal because of the duality between $\mathcal{F}(\Delta^A)$ and $\mathcal{F}(\Delta^B)$.

Condition (1) is equivalent to the statement that $\Delta^A_{H,V}$ is Ext-injective in $\mathcal{F}(\Delta^A)$, because the indecomposable Ext-injectives are precisely the $T^A_{K,W}$, and $T^A_{H,V}$ is the Ext-injective hull of $\Delta^A_{H,V}$ (in the terminology of [29]). Since injectives are Ext-injective, we obtain the implication (2) $\Rightarrow$ (1).

We next prove (4) $\Rightarrow$ (2), namely that if $H = 1$ then $\Delta^A_{H,V}$ is injective. We may construct injectives using the duality between covariant functors and contravariant functors on $\mathcal{O}_\mathcal{S}$. Writing $M(x)^* = \mathrm{Hom}_k(M(x), k)$, given a covariant functor $M$ we obtain a contravariant functor $M^*$, and conversely, and it is evident that projectives and injectives are interchanged by this duality. Since $k\,\mathrm{Hom}(-, x)$ is a projective contravariant functor (by an analogous result to Proposition 2.3), the functor $k\,\mathrm{Hom}(-, x)^*$ is an injective covariant functor. Taking $x$ to be the object $G/1$ in $\mathcal{O}_\mathcal{S}$ we obtain an injective functor which is the direct sum of the $\Delta^A_{1,V}$ corresponding to the decomposition of the regular representation $kG$ into projectives. Since every $\Delta^A_{1,V}$ appears here, they are all injective.

We prove finally that (1) $\Rightarrow$ (4) and will do it by showing that if $H \neq 1$ is a $p$-subgroup and $V$ is a simple $k[N_G(H)/H]$-module then there is a non-split short exact sequence of $k\mathcal{O}_\mathcal{S}$-modules

$$0 \to \Delta^A_{H,V} \to M_1 \to M_2 \to 0$$

in which all terms lie in $\mathcal{F}(\Delta^A)$. This will show that $\Delta^A_{H,V}$ is not Ext-injective in $\mathcal{F}(\Delta^A)$ and hence not $T^A_{H,V}$.

We will construct $M_1$ in stages. Let $X$ be the projective cover of $V$ as a $k[N_G(H)/H]$-module. When we regard $X$ as a $k[N_G(H)]$-module it has vertex $H$ and a Green correspondent $U$, say, as a $kG$-module (see [4] for background). Consider the $k\mathcal{O}_\mathcal{S}$-module $F = FQ_U$ which assigns to each subgroup $J$ the fixed quotient $FQ_U(J) = k \otimes_{kJ} U$. Now $F(1) = U$ is indecomposable and the functorial morphisms $F(1) \to F(H)$ (i.e. the corestriction maps) are all surjective, so $F$ is generated as a $k\mathcal{O}_\mathcal{S}$-module by $F(1)$. It follows that $F$ is indecomposable as a $k\mathcal{O}_\mathcal{S}$-module, since if $F = F_1 \oplus F_2$ then one summand, say $F_1$, must have $F_1(1) = F(1)$ and hence $F_1 = F$, because $F$ is generated by its value at 1.

Next we claim that the value of $F$ at $H$ has $X$ as a summand. This is because the restriction $U\downarrow^G_{N_G(H)}$ has $X$ as a summand by the properties of Green correspondence, and this survives to the fixed quotient under $H$. Thus $\Delta^A_{H,V}$ is a quotient of a submodule of $F$, because it is supported only at $H$.

Now let $L \to F$ be the projective cover of $F$ as a $k\mathcal{O}_\mathcal{S}$-module, so that $L$ is also generated by its value at 1, and $L \cong P^A_{1,V_1} \oplus \cdots \oplus P^A_{1,V_n}$ as a direct sum of indecomposable projectives. For each subgroup $Q$ we let $\langle L(Q)\rangle$ be the subfunctor of $L$ generated by its value at $Q$. Now

$$\sum_{Q \geq H} \langle L(Q)\rangle / \sum_{Q>H} \langle L(Q)\rangle \cong \bigoplus \Delta^A_{H,V_i}$$

has $\Delta^A_{H,V}$ as an image, and so some indecomposable projective functor $P^A_{1,W}$ has $\Delta^A_{H,V}$ appearing in a $\Delta^A$-filtration. The quotient by the term in this filtration of $P^A_{1,W}$ which leaves $\Delta^A_{H,V}$ in the socle of the quotient is an indecomposable module $M$ in $\mathcal{F}(\Delta^A)$ (indecomposable because its semisimple quotient is simple) which, provided $H \neq 1$, gives rise to a non-split extension $0 \to \Delta^A_{H,V} \to M_1 \to M_2 \to 0$ with all terms in $\mathcal{F}(\Delta^A)$. It follows if $H \neq 1$ that $\Delta^A_{H,V}$ is not Ext-injective. □

Although we will not use it in reformulating Alperin's conjecture, we include the next result for completeness.

**Theorem 4.3.** *Let $A = k\mathcal{O}_\mathcal{S}$ where $\mathcal{O}_\mathcal{S}$ is the orbit category of $G$ with respect to its $p$-subgroups, and let $B$ be the Ringel dual of $A$. The following are equivalent.*

1. *$\Delta^A_{H,V} = P^A_{H,V}$ is a projective $k\mathcal{O}_\mathcal{S}$-module,*
2. *$\Delta^A_{H,V}$ is Ext-projective in $\mathcal{F}(\Delta^A)$,*
3. *$\Delta^B_{H,V} = T^B_{H,V}$ is a canonical $B$-module,*
4. *$H$ is a Sylow $p$-subgroup of $G$.*

*Proof.* Some of the implications are immediate. We have (4) $\Rightarrow$ (1) because $(H, V)$ is a minimal element of $\Lambda$. Also, (1) $\Rightarrow$ (2) is trivial. (2) $\Leftrightarrow$ (3) follows from the duality between $\mathcal{F}(\Delta^A)$ and $\mathcal{F}(\Delta^B)$.

(2) $\Rightarrow$ (1) holds because the projectives lie in $\mathcal{F}(\Delta^A)$, so they lie among the Ext-projective objects, and the number of indecomposable Ext-projectives equals the number of indecomposable projectives (for example, by [29, 8.6] where it is shown that the indecomposable Ext-projectives are the Ext-projective covers of the $\Delta$s).

It remains to demonstrate (1) $\Rightarrow$ (4), which we do by showing that if $H$ is not a Sylow $p$-subgroup then $\Delta^A_{H,V}$ is not projective. So let $H$ be a $p$-subgroup of $G$ and let $X$ be the projective cover of $V$ as a $k[N_G(H)/H]$-module. In the notation used in [27] and [29], consider the Mackey functor $(\mathrm{Inf}^{N_G(H)}_{N_G(H)/H} FQ_X) \uparrow^G_{N_G(H)}$, namely the inflation to $N_G(H)$ of the fixed quotient functor $FQ_X$, induced to $G$. In [29] such Mackey functors played a key role as factors in certain filtrations, and were denoted there $\Delta_{H,X}$, but to avoid confusion we do not use this notation here, because it is not the same as our present usage. In [29, 3.4] we show that this Mackey functor is generated by its value at $H$. This implies that it is still generated by its value at $H$ if we consider only the operations which form the covariant part of the Mackey functor (see e.g. [26, 2.1]), and hence if we let $M$ be the $k\mathcal{O}_{\mathcal{S}}$-module which is the restriction to $\mathcal{O}_{\mathcal{S}}$ of this Mackey functor then $M$ is generated by its value at $H$. As in [29, 3.1] we have $M(H) = X$, which is indecomposable, and so we deduce that $M$ is indecomposable as an $k\mathcal{O}_{\mathcal{S}}$-module (since if $M = M_1 \oplus M_2$ then, say, $M_1(H) = M(H)$ and hence $M_1 = M$).

Consider now the short exact sequence of $k\mathcal{O}_{\mathcal{S}}$-modules

$$0 \to K \to M \to \Delta^A_{H,V} \to 0$$

where $K$ is the submodule of $M$ generated by the values of $M$ at groups strictly larger than $H$. Suppose that $H$ is not a Sylow $p$-subgroup of $G$. Then there is a non-identity $p$-subgroup $J/H$ of $N_G(H)/H$ and it has a non-zero fixed quotient $X_J$. This implies that $M(J) \neq 0$ and hence that $K \neq 0$ (because $K(J) \neq 0$). Since $M$ is indecomposable the short exact sequence is not split, and so $\Delta^A_{H,V}$ is not projective. □

We immediately obtain several reformulations of Alperin's weight conjecture by substituting for the numbers of simple $kG$-modules or weights the other numbers indicated by these theorems. The reformulations are appealing because they are expressed in terms of objects defined by categorical properties of a single category, $k\mathcal{O}_{\mathcal{S}}$-mod, together with knowledge of the preorder on $\Lambda$. The structure of this category has the potential to provide a reason why the numbers are equal since up to Morita equivalence the group algebra may be recovered from $k\mathcal{O}_{\mathcal{S}}$-mod and $\Lambda$ using Theorem 4.2. Our hope is that the theory of stratifications may shed light on Alperin's conjecture.

Some of our reformulations of Alperin's conjecture reveal a symmetry between the two numbers which it asserts to be equal, and it is particularly apparent if we allow ourselves to express them in terms of the Ringel dual. We single out now two of the reformulations which have this symmetry.

**Corollary 4.4.** *Let $A = k\mathcal{O}_{\mathcal{S}}$ where $\mathcal{O}_{\mathcal{S}}$ is the orbit category of $G$ with respect to its $p$-subgroups, and let $B$ be the Ringel dual of $A$. The assertion of Alperin's weight conjecture for the group $G$ is equivalent to each of the following two statements.*

(1) *The number of $\lambda = (H, V)$ for which $\Delta^A_\lambda = T^A_\lambda$ is equal to the number of $\lambda$ for which $\overline{\nabla}^B_\lambda = T^B_\lambda$.*

| $\lambda$ | $(1,1)$ | $(1,2)$ | $(C_2,1)$ |
|---|---|---|---|
| simple $S_\lambda^A$ | $a$ | $b$ | $c$ |
| projective $P_\lambda^A$ | $\begin{smallmatrix} & a & \\ a & & c \end{smallmatrix}$ | $\begin{smallmatrix} b \\ c \end{smallmatrix}$ | $c$ |
| injective $I_\lambda^A$ | $\begin{smallmatrix} a \\ a \end{smallmatrix}$ | $b$ | $\begin{smallmatrix} a & & b \\ & c & \end{smallmatrix}$ |
| $\Delta_\lambda^A$ | $\begin{smallmatrix} a \\ a \end{smallmatrix}$ | $b$ | $c$ |
| $\overline{\nabla}_\lambda^A$ | $a$ | $b$ | $\begin{smallmatrix} a & & b \\ & c & \end{smallmatrix}$ |
| $T_\lambda^A$ | $\begin{smallmatrix} a \\ a \end{smallmatrix}$ | $b$ | $\begin{smallmatrix} & a & & b \\ a & & c & \end{smallmatrix}$ |

TABLE 1. Modules for $A = \mathbb{F}_2\mathcal{O}_\mathcal{S}$ where $\mathcal{S}$ is the set of 2-subgroups of $S_3$.

(2) *The number of $\lambda = (H, V)$ for which $\Delta_\lambda^B = P_\lambda^B$ is equal to the number of $\lambda$ for which $\overline{\nabla}_\lambda^A = I_\lambda^A$.*

We illustrate these assertions with a small example, which nevertheless demonstrates the key features. We consider the orbit category $\mathcal{O}_\mathcal{S}$ for the symmetric group $G = S_3$, taking $\mathcal{S}$ to be the 2-subgroups of $S_3$. Thus $\mathcal{S}$ contains the identity subgroup and three subgroups of order 2 and $\mathcal{O}_\mathcal{S}$ has four objects. Since the three subgroups of order 2 are all conjugate, the corresponding objects of $\mathcal{O}_\mathcal{S}$ are all isomorphic, and we denote one of these objects by $G/C_2$. By Lemma 2.1 we can delete the other two isomorphic objects without changing the Morita type of the category algebra, and we do this. The automorphism groups of the objects are $\mathrm{Aut}(G/1) \cong G$ and $\mathrm{Aut}(G/C_2) = 1$, so that taking $k = \mathbb{F}_2$ (a splitting field!) there are three simple representations of $\mathcal{O}_\mathcal{S}$ parametrized by pairs $(G/1, 1) = (1, 1)$, $(G/1, 2) = (1, 2)$ and $(G/C_2, 1) = (C_2, 1)$, where we denote representations of the automorphism groups by their dimensions. To simplify the notation we will write the simple $k\mathcal{O}_\mathcal{S}$-modules as $a = S_{1,1}^A$, $b = S_{1,2}^A$ and $c = S_{C_2,1}^A$.

The relevant representations of $A = k\mathcal{O}_\mathcal{S}$ and its Ringel dual, $B$, appear in Tables 1 and 2. We see that the weights $(1, 2)$ and $(C_2, 1)$ determine the columns of the table for $A$ in which the $\Delta_\lambda^A$ are simple, and also in which the $\overline{\nabla}_\lambda^A$ are injective. In the table for $B$ these are the columns in which $\overline{\nabla}_\lambda^B = T_\lambda^B$. At the same time the columns indexed by $(1, 1)$ and $(1, 2)$ are those in which $\Delta_\lambda^A = T_\lambda^A$ or equivalently $\Delta_\lambda^A = I_\lambda^A$ for $A$, and $P_\lambda^B = \Delta_\lambda^B$ for $B$. For completeness we describe all the indecomposable $\mathbb{F}_2\mathcal{O}_\mathcal{S}$-modules. The ones in $\mathcal{F}(\Delta^A)$ and $\mathcal{F}(\overline{\nabla}^A)$ appear

| $\lambda$ | $(1,1)$ | $(1,2)$ | $(C_2,1)$ |
|---|---|---|---|
| simple $S^B_\lambda$ | $\alpha$ | $\beta$ | $\gamma$ |
| projective $P^B_\lambda$ | $\begin{smallmatrix}\alpha\\ \gamma\\ \alpha\end{smallmatrix}$ | $\beta$ | $\begin{smallmatrix} & \gamma & \\ \alpha & & \beta\\ \gamma & & \\ \alpha & & \end{smallmatrix}$ |
| injective $I^B_\lambda$ | $\begin{smallmatrix}\gamma\\ \alpha\\ \gamma\\ \alpha\end{smallmatrix}$ | $\begin{smallmatrix}\gamma\\ \beta\end{smallmatrix}$ | $\begin{smallmatrix}\gamma\\ \alpha\\ \gamma\end{smallmatrix}$ |
| $\Delta^B_\lambda$ | $\begin{smallmatrix}\alpha\\ \gamma\\ \alpha\end{smallmatrix}$ | $\beta$ | $\gamma$ |
| $\overline{\nabla}^B_\lambda$ | $\begin{smallmatrix}\gamma\\ \alpha\end{smallmatrix}$ | $\begin{smallmatrix}\gamma\\ \beta\end{smallmatrix}$ | $\gamma$ |
| $T^B_\lambda$ | $\begin{smallmatrix}\gamma\\ \alpha\\ \gamma\\ \alpha\end{smallmatrix}$ | $\begin{smallmatrix}\gamma\\ \beta\end{smallmatrix}$ | $\gamma$ |

TABLE 2. Modules for $B =$ the Ringel dual of $A$.

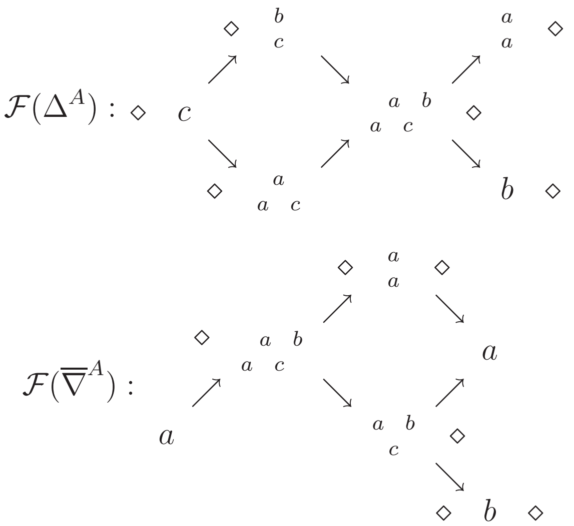


TABLE 3. Modules for $A = \mathbb{F}_2\mathcal{O}_\mathcal{S}$ with a $\Delta^A$-filtration and with a $\overline{\nabla}^A$-filtration.

in Table 3. In addition to the modules displayed in these diagrams there is one further indecomposable $\mathbb{F}_2\mathcal{O}_\mathcal{S}$-module, which is $\begin{smallmatrix}a\\ c\end{smallmatrix}$. In these pictures we place a symbol $\diamond$ to the left of a module when it is Ext-projective, and to the right of a module when it is Ext-injective.

We conclude with some remarks. The first is that there is a block-by-block refinement of Alperin's conjecture which may be described as follows. Given a block idempotent $e^2 = e \in Z(kG)$, for each $p$-subgroup $H \leq G$ the Brauer morphism provides a central idempotent $\mathrm{Br}_H(e) \in Z(k[C_G(H)]) \subseteq k[N_G(H)]$ (see [25]). Regarding each simple $k[N_G(H)/H]$-module $V$ as a $k[N_G(H)]$-module we may say that $V$ belongs to $e$ if $\mathrm{Br}_H(e) \cdot V \neq 0$ (in which case $\mathrm{Br}_H(e)$ acts as the identity on $V$ and $\mathrm{Br}_H(e) \cdot V = V$). The block-by-block version of Alperin's weight conjecture states that for each block of $kG$, the number of simple $kG$-modules belonging to the block equals the number of weights belonging to the block.

We see from the example with $S_3$ at $p = 2$ that $k\mathcal{O}_\mathcal{S}$ need not respect the blocks of $kG$ since $kS_3$ has two blocks, but $k\mathcal{O}_\mathcal{S}$ has only one block. It thus appears at first sight that the reformulations of Alperin's conjecture using $k\mathcal{O}_\mathcal{S}$ might not be well adapted to block-by-block versions.

In fact, we may produce such reformulations in the following way. The category algebra $k\mathcal{O}_\mathcal{S}$ contains an idempotent $1_{G/H}$ for each object $G/H$ of $\mathcal{O}_\mathcal{S}$ These idempotents are orthogonal and $1_{G/H} \cdot k\mathcal{O}_\mathcal{S} \cdot 1_{G/H} \cong k[N_G(H)/H]$. Given a block idempotent $e$ of $kG$ we regard each $\mathrm{Br}_H(e)$ as an element of $1_{G/H} \cdot k\mathcal{O}_\mathcal{S} \cdot 1_{G/H}$ and let $E = \sum_{H \in \mathcal{S}/G} \mathrm{Br}_H(e)$. Now $E$ is idempotent, and $E \cdot k\mathcal{O}_\mathcal{S} \cdot E$ is an algebra whose simple modules are the $S_{H,V}$ where $V$ belongs to $e$, and which is standardly stratified, with structure obtained by applying the functor $M \to E \cdot M$ to the stratification of $k\mathcal{O}_\mathcal{S}$. We leave it to the interested reader to complete the argument that the analogues of 4.1 and 4.2 hold for $E \cdot k\mathcal{O}_\mathcal{S} \cdot E$. Thus, for example, $H = 1$ and $V$ belongs to $e$ if and only if $\Delta^{EAE}_{H,V} = T^{EAE}_{H,V}$; and $(H, V)$ is a weight belonging to $e$ if and only if $\overline{\nabla}^{EAE}_{H,V} = I^{EAE}_{H,V}$.

We remark that the property required of $k\mathcal{O}_\mathcal{S}$ which Corollary 4.4 states to be equivalent to Alperin's conjecture is not satisfied by standardly stratified algebras in general, and it is quite easy to find examples of this. One such example is given on page 153 of [2] where there is exhibited a standardly stratified algebra in which the number of $\lambda$ for which $\Delta_\lambda = T_\lambda$ is 1 but the number of $\lambda$ for which $\overline{\nabla}_\lambda = I_\lambda$ is 0. This means that to establish Alperin's conjecture by using these reformulations, further specific properties of the orbit category algebra $k\mathcal{O}_\mathcal{S}$ must be used. If we knew how to identify the right properties then it would no doubt be possible to prove Alperin's conjecture. Equally, without knowing a proof it is hard to see what the appropriate properties might be.

The many reformulations of the conjecture which have appeared over the years can be viewed as attempts to identify the mechanism which underlies it, and the present work is no different in this respect. Like some other attempts we have focused on the structure of the orbit category, and we mention [19, 24] as recent contributions which also exploit this structure. By comparison, another recent approach of Linckelmann [18] uses the structure of the Frobenius category. This has the advantage that it behaves better than the orbit category when it comes to getting block-by-block versions of the conjecture, but it has the disadvantage that the Frobenius category loses some information about $p$-subgroups, which it appears to be necessary to restore by considering an appropriate extension category. At the moment it is hard to know which category to favour. It seems quite possible that once the conceptual underpinnings of Alperin's conjecture have been properly worked out and understood, many of these approaches to the conjecture will become viable.

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## References

[1] I.Ágoston, V. Dlab, E. Lukács, *Stratified algebras*, C. R. Math. Acad. Sci. Soc. R. Can. 20 (1998), 22-28.

[2] I.Ágoston, D. Happel, E. Lukács, L. Unger, *Standardly stratified algebras and tilting,* J. Algebra 226 (2000), 144-160.

[3] J.L. Alperin, *Weights for finite groups,* Proc. Symp. Pure Math. 47 (1987), 369-379.

[4] J.L. Alperin, *Local representation theory,* Cambridge University Press 1993.

[5] M. Auslander and I. Reiten, *Applications of contravariantly finite subcategories,* Adv. Math. 86 (1991), 111-152.

[6] G.E. Bredon, *Equivariant cohomology theories,* Lecture Notes in Math. 34, Springer-Verlag 1967.

[7] C. Broto, R. Levi, B. Oliver, *The theory of p-local groups: a survey,* Homotopy theory: relations with algebraic geometry, group cohomology and algebraic $K$-theory 51 - 84, Contemp. Math. 346, AMS Providence R.I. 2004.

[8] E. Cline, B. Parshall and L. Scott, *Stratifying endomorphism algebras,* Mem. Amer. Math. Soc. 124 (1996), no. 591.

[9] T. tom Dieck, *Transformation Groups*, Walter de Gruyter, Berlin - New York 1987.

[10] V. Dlab, *Quasi-hereditary algebras revisited,* An. S. Univ. Ovidius Constantza 4 (1996), 43-54.

[11] V. Dlab and C.M. Ringel, *The module theoretical approach to quasi-hereditary algebras,* in 'Representations of Algebras and Related Topics,' (H. Tachikawa and S. Brenner, eds.), pp. 200-224. London Mathematical Society Lecture Note Series, vol 168 (1992).

[12] W.G. Dwyer and H.-W. Henn, *Homotopy theoretic methods in group cohomology,* Advanced courses in Math. CRM Barcelona, Birkhäuser Verlag, Boston 2001.

[13] A. Frisk, *Dlab's theorem and tilting modules for stratified algebras,* preprint (2004) obtainable from http://www.math.uu.se/research/pub/preprints.php

[14] D. Happel, *Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras,* London Mathematical Society Lecture Note Series, vol. 119 (1988).

[15] S. Jackowski, J. McClure, B. Oliver, *Homotopy classification of self-maps of BG via G-actions,* Annals of Math. 135 (1992), 183-270.

[16] S. Koenig, *Filtrations, stratifications and applications,* 65-111 in ed. V. Dlab and C.M. Ringel, Representations of finite dimensional algebras and related topics in Lie theory and geometry, Fields Inst Comm. 40 (2004).

[17] P. Lakatos, *On a theorem of V. Dlab,* Algebras and Repn. Thy. 3 (2000), 99-103.

[18] M. Linckelmann, *Fusion category algebras,* J. Algebra 277 (2004), 222-235.

[19] M. Linckelmann, *Alperin's weight conjecture in terms of equivariant Bredon cohomology,* Math. Z. 250 (2005), 495-513.

[20] W. Lück, *Transformation groups and algebraic $K$-theory,* Lecture Notes in Math. 1408, Springer-Verlag, Berlin 1989.

[21] Y. Miyashita, *Tilting modules of finite projective dimension,* Math. Z. 193 (1986), 113-146.

[22] B. Oliver, *Higher limits via Steinberg representations,* Comm. Algebra 22 (1994), 1381-1393.

[23] C.M. Ringel, *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences,* Math. Zeit. 208 (1991), 209-225.

[24] P. Symonds, *The Bredon cohomology of subgroup complexes,* J. Pure Appl. Algebra, 199 (2005), 261-298.

[25] J. Thévenaz, *G-Algebras and modular representation theory,* Oxford University Press 1995.

[26] J. Thévenaz and P.J. Webb, *Simple Mackey functors,* in: Proc. of 2nd international group theory conference, Bressanone (1989), Supplemento ai Rendiconti del Circolo Matematico di Palermo 23 (1990), 299-319.

[27] J. Thévenaz and P.J. Webb, *The structure of Mackey functors,* Trans. Amer. Math. Soc. 347 (1995), 1865-1961.

[28] P.J. Webb, *A guide to Mackey functors*, M. Hazewinkel (ed.), Handbook of Algebra vol. 2, Elsevier 2000, pp. 805-836.

[29] P.J. Webb, *Stratifications and Mackey functors I: functors for a single group,* Proc. London Math. Soc. (3) 82 (2001), 299-336.

[30] C. Xi, *Standardly stratified algebras and cellular algebras,* Math. Proc. Camb. Phil. Soc. 133 (2002), 37-53.

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