A Guide to Mackey Functors

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Contents:

1. Introduction

A Mackey functor is an algebraic structure possessing operations which behave like the induction, restriction and conjugation mappings in group representation theory. Operations such as these appear in quite a variety of diverse contexts — for example group cohomology, the algebraic K-theory of group rings, and algebraic number theory — and it is their widespread occurrence which motivates the study of such operations in abstract.

The axioms for a Mackey functor which we will use were first formulated by Dress [24], [25] and by Green [30]. They follow on from earlier ideas of Lam on Frobenius functors [36], described in [19]. Another structure which appeared early on is Bredon's notion of a coefficient system [15].

A major preoccupation in studying Mackey functors is to compute their values, be it in the context of specific examples such as computing the cohomology or character ring of a finite group, or in a more general setting. It is important to develop techniques to do this, and if some method of calculation can be formulated within the general context of Mackey functors then we have the possibility to apply it to every specific instance without developing it each time from scratch.

One argument which generalizes to Mackey functors in this way is the method of stable elements which appears in the book of Cartan and Eilenberg [16], and which provides a way of computing the p-torsion subgroup of the cohomology of a finite group as a specifically identified subset of the cohomology of a Sylow p-subgroup. An ingredient in the general form of this calculation is the notion of relative projectivity of a Mackey functor, similar in spirit to the notion of relative projectivity of group representations. We will see in Section 3 how the method of stable elements can be formulated for all Mackey functors.

Induction theorems are another kind of result which are among the most important methods of computation. They have a very well-developed and well-known theory, especially in the context of group representations. Such theorems may also be formulated in the general setting, and in Section 6 we present an important induction theorem due to Dress. Work of great refinement obtaining explicit forms of induction theorems has been done by Boltje [5], [6], [7], but this goes beyond what we can describe here.

In order to present these applications we develop the technical machinery which they necessitate, and we do this in an order which to a large extent reflects chronology. As the theory of Mackey functors became more elaborate it became apparent that they are algebraic structures in their own right with a theory which fits into the framework of representations of algebras. They may, in fact, be identified with the representations of a certain algebra — called the Mackey algebra — and there are simple Mackey functors, projective and injective Mackey functors, resolutions of Mackey functors, and so on. We describe this theory in outline in Section 5.

A new notion of Mackey functor began to appear, namely that of a globally-defined Mackey functor, an early instance of which appeared in the work of Symonds [52]. These are structures which have a definition on all finite groups (whereas the original Mackey functors are only defined on the subgroups of some fixed group) and we present in Section 8 the context for these structures envisaged by Bouc [10]. We describe three uses for these functors: a method of computing group cohomology in Section 9, an approach to the stable decomposition of classifying spaces BG in Section 10, and a framework in which Dade's group of endopermutation modules plays a fundamental role in Section 11.

There is no full account of Mackey functors in text book form, and with this in mind I have tried to be comprehensive in my treatment. In this I have failed, and on top of everything the proofs that are given are often sketchy or left to the reader who must either work them out as an exercise or consult the literature. This guide to Mackey functors is deliberately concise. The omissions which seem most regrettable are these: the definition of a Mackey functor on compact Lie groups and other more general classes of groups (see [38], [21]); the theory of Green functors (see [59], [12]); and the theory of Brauer quotients (see [59]).

Finally, I wish to thank Serge Bouc for his comments on this exposition.

2. The definitions of a Mackey functor

There are several ways of giving the definition of a Mackey functor, but they all amount to the same thing. We present two definitions here, the first in terms of many axioms and the second in terms of bivariant functors on the category of finite G-sets. They may also be defined as functors on a specially-constructed category, an approach which is due to Lindner [40].

The most accessible definition of a Mackey functor for a finite group G is expressed in terms of axiomatic relations. We fix a commutative ring R with a 1 and let R-mod denote the category of R-modules. A *Mackey functor* for G over R is a function

$$M : \{\text{subgroups of } G\} \to R\text{-mod}$$

with morphisms

$$\begin{aligned} I_K^H &: M(K) \to M(H) \\ R_K^H &: M(H) \to M(K) \\ c_g &: M(H) \to M({}^gH) \end{aligned}$$

for all subgroups H and K of G with $K \leq H$ and for all g in G, such that

(0) $I_H^H, R_H^H, c_h : M(H) \to M(H)$ are the identity morphisms for all subgroups H and $h \in H$

(1) $R_J^K R_K^H = R_J^H$ } for all subgroups $J \leq K \leq H$

(2) $I_K^H I_J^K = I_J^H$ } for all subgroups $J \leq K \leq H$

(3) $c_g c_h = c_{gh}$ for all $g, h \in G$

(4) $R^{{}^gH}_{{}^gK} c_g = c_g R^H_K$ } for all subgroups $K \le H$ and $g \in G$
(5) $I^{{}^gH}_{{}^gK} c_g = c_g I^H_K$ }
(6) $R^H_J I^H_K = \sum_{x \in [J\backslash H/K]} I^J_{J \cap {}^xK} c_x R^K_{J^x \cap K}$ for all subgroups $J, K \le H$.

We use the letters I, R and c because these operations are reminiscent of induction, restriction and conjugation of characters. We should properly write $c_{g,H}$ instead of c_g, since our notation does not distinguish between the conjugation morphisms with the same element g but different domain $M(H)$. The most elaborate of these axioms is (6), which is called the *Mackey decomposition formula,* and is responsible for the name of these functors. It is familiar from representation theory and cohomology. In this axiom we are using the notation $[J\backslash H/K]$ to denote a set of representatives in G for the double cosets $J\backslash H/K$. We write ${}^xH = xHx^{-1}$ and $H^x = x^{-1}Hx$.

Mackey functors form a category denoted $\mathrm{Mack}_R(G)$ in which the morphisms are natural transformations of Mackey functors; that is, a morphism $\eta : M \to N$ is a family of R-module homomorphisms $\eta_H : M(H) \to N(H)$ commuting with all operations I, R and c. This category is abelian, the reason being that R-mod is abelian, and in fact we may define kernels, cokernels, subfunctors, quotient functors and so forth pointwise using the fact that they exist in R-mod. We may speak of the intersection of subfunctors of a Mackey functor, defined pointwise, and it is again a subfunctor. If we are given for each subgroup $H \le G$ a subset $N(H) \subseteq M(H)$ we may speak of the subfunctor $\langle N \rangle$ generated by N: it is the intersection of the subfunctors containing N.

We will encounter also the notion of a *Green functor*, which is a Mackey functor M with an extra multiplicative structure. Specifically, for each subgroup $H \le G$, $M(H)$ should be an associative R-algebra with identity so that

(7) the R^H_K and c_g are always unitary R-algebra homomorphisms, and
(8) for all subgroups $K \le H$, $a \in M(K)$ and $b \in M(H)$ we have
$I^H_K(a \cdot R^H_K(b)) = I^H_K(a) \cdot b$ and $I^H_K(R^H_K(b) \cdot a) = b \cdot I^H_K(a)$

Axiom (8) is called the Frobenius axiom. Green functors have in some ways a tighter structure than Mackey functors, but we will not describe their theory in detail here. Fuller accounts of recent theory may be found in [58], [59] and [12].

We mention now some examples of Mackey functors that immediately come to mind. We may take $M(G)$ to be

- $G_0(kG)$: the Grothendieck group of the category of finitely generated kG-modules. In characteristic zero this may be identified as the group of characters of kG-modules, and in characteristic p as the group of Brauer characters.
- $A(G)$: the Green ring of finitely generated kG-modules [19, Sect. 81].
- $H^n(G, U)$, $H_n(G, U)$, $\hat{H}^n(G, U)$: the cohomology, homology and Tate cohomology of G in some dimension n with coefficients in the $\mathbb{Z}G$-module U.
- $B(G)$: the Burnside ring of G. We may identify this as the free abelian group with the isomorphism types of transitive G-sets as a basis.

- $K_n(\mathbb{Z}G)$: the algebraic K-theory of $\mathbb{Z}G$, and other related groups such as the Whitehead group.
- $\mathrm{Cl}(\mathcal{O}(F^G))$: the class group of the ring of integers of the fixed field F^G where G is a group of automorphisms of a number field F (see [35], [50], [8]).

For some more examples see [59, Sect. 53].

In the first instance these examples are only Mackey functors over the ground ring $\mathbb{Z}$. If we have some other ground ring R in mind we may always form a Mackey functor $R \otimes_{\mathbb{Z}} M$ whose values are $R \otimes_{\mathbb{Z}} M(H)$ for each subgroup H of G. Some examples may already be naturally defined over a ring R other than $\mathbb{Z}$, for example Tate cohomology. Since $|G|$ annihilates Tate cohomology, this example gives a Mackey functor over $\mathbb{Z}/|G|\mathbb{Z}$; if the module U happens to be defined over some further ring R then $M(G) = H^n(G, U)$ is also a Mackey functor over R.

It is important to have available a different definition of Mackey functors, less dependent on a large number of axioms. It is phrased in terms of the category G-set whose objects are the finite left G-sets, and whose morphisms are the G-equivariant mappings. We will be especially interested in the space of left cosets G/H for each subgroup H of G: each G-set is isomorphic to a disjoint union of these. We may now define a Mackey functor over R to be a pair of functors $M = (M_*, M^*)$ from G-set to R-mod so that M_* is covariant, M^* is contravariant, $M_*(\Omega) = M^*(\Omega)$ for all finite G-sets Ω, and such that the following axioms are satisfied:

(1) for every pullback diagram of G-sets

$$\begin{array}{ccc} \Omega_1 & \xrightarrow{\alpha} & \Omega_2 \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ \Omega_3 & \xrightarrow{\delta} & \Omega_4 \end{array}$$

we have $M^*(\delta)M_*(\gamma) = M_*(\beta)M^*(\alpha)$, and

(2) for every pair of finite G-sets Ω and Ψ, applying M_* to $\Omega \to \Omega \sqcup \Psi \leftarrow \Psi$ gives the component maps in a morphism $M(\Omega) \oplus M(\Psi) \to M(\Omega \sqcup \Psi)$, which we require to be an isomorphism.

The definition we have just given is a special case of the definition given by Dress [25], who phrased it in terms of more general categories than G-set and R-mod, given by certain axioms.

To some extent the definition of Mackey functors in terms of G-sets is a question of notation: according to the first definition we would write $M(H)$ for the value of M at the subgroup H, whereas with the G-set definition we would write $M(G/H)$. In this account we will sometimes use both of these notations and switch from one to the other without special comment. It should be clear by looking at whether the argument is a subgroup or a G-set which notation we are using.

To make the connection between the two definitions on morphisms, we first identify two particular morphisms of G-sets. When H is a subgroup of K there is a morphism $\pi_H^K : G/H \to G/K$ specified by $\pi_H^K(xH) = xK$. When $g \in G$ and H is a subgroup of G there is also a morphism $c_g : G/H \to G/{}^gH$ specified by $c_g(xH) = xg^{-1}\,{}^gH$. It is the case that any morphism between coset spaces is a composite of these two types of morphisms (and it is an instructive exercise to prove it).

We now identify the operation I_H^K of a Mackey functor given according to the first definition with the morphism $M_*(\pi_H^K)$, and R_H^K with $M^*(\pi_H^K)$. The operation c_g of the first definition is identified with $M_*(c_g)$, which is necessarily equal to $M^*(c_{g^{-1}})$ in the presence of the axioms.

We have to check that the axioms of the first definition imply the axioms of the second, and vice-versa. Most of this is routine, and the most sophisticated aspect is the reformulation of the Mackey formula as the axiom on pullbacks. The key here is the following result.

(2.1) LEMMA. *Whenever H and K are subgroups of J, itself a subgroup of G, there is a pullback diagram of G-sets*

$$\begin{array}{ccc} \Omega & \longrightarrow & G/K \\ \big\downarrow & & \big\downarrow \\ G/H & \longrightarrow & G/J \end{array}$$

where $\Omega = \bigsqcup_{x\in[H\backslash J/K]} G/(H \cap {}^xK)$. With this identification of Ω the map $\Omega \to G/H$ has components $\pi^H_{H\cap {}^xK}$, and $\Omega \to G/K$ has components $\pi^K_{H^x\cap K}c_{x^{-1}}$.

Proof. We first observe that

$$\begin{array}{ccc} J/H \times J/K & \longrightarrow & J/K \\ \big\downarrow & & \big\downarrow \\ J/H & \longrightarrow & J/J \end{array}$$

is a pullback diagram and that $J/H \times J/K \cong \bigsqcup_{x\in[H\backslash J/K]} J/(H \cap {}^xK)$. Now apply induction of G-sets from J to G (defined in the next section) to this diagram. □

In view of this it is immediate that the pullback axiom of the second definition implies the Mackey decomposition formula of the first. Conversely, the Mackey decomposition formula implies the pullback axiom for pullbacks of this form, and this is in fact sufficient to imply the axiom for all pullbacks.

3. The computation of Mackey functors using relative projectivity

The basic notion which permits the computation of a Mackey functor along the lines of the Cartan-Eilenberg stable elements method is that of relative projectivity, which is formally similar to relative projectivity in the context of representation theory. It may be expressed most intuitively in terms of induction and restriction.

We define *induction* and *restriction* of Mackey functors in terms of induction and restriction of G-sets. If Ω is a G-set and H a subgroup of G then $\Omega \downarrow_H^G$ denotes the set Ω regarded as an H-set by restriction of the action. If Ψ is an H-set we define a G-set $\Psi \uparrow_H^G = G \times_H \Psi$, namely the equivalence classes in $G \times \Psi$ of the equivalence relation $(gh, \psi) \sim (g, h\psi)$ whenever $g \in G$, $h \in H$ and $\psi \in \Psi$. Another way to describe this is that it is the set of orbits under the action of H on $G \times \Psi$ given by $h(g, \psi) = (gh^{-1}, h\psi)$ where $g \in G$, $h \in H$ and $\psi \in \Psi$. The action of G on $G \times_H \Psi$ comes from the left multiplication of G on G. We may check that $\uparrow_H^G$ is left adjoint to $\downarrow_H^G$ (but it is not right adjoint in general). We now define restriction and induction of Mackey functors by

$$N \uparrow_H^G (\Omega) = N(\Omega \downarrow_H^G)$$
$$M \downarrow_H^G (\Psi) = M(\Psi \uparrow_H^G).$$

Restriction of Mackey functors is what we would expect: regarding M as being defined on subgroups of G, if $K \leq H$ then $M \downarrow_H^G (K) = M(K)$. Induction is more complicated, and for subgroups H, K of G there is a formula

$$N \uparrow_H^G (K) = \bigoplus_{g \in [H \backslash G / K]} N(H \cap {}^g K).$$

Induction and restriction of Mackey functors satisfy relationships inherited from the corresponding operations for G-sets and most of them are what we would expect; for example there is a Mackey decomposition formula for $M \uparrow_K^G \downarrow_J^G$. The property which is perhaps surprising is that induction of Mackey functors is both left and right adjoint to restriction. A formal consequence of this, which we mention now and will use later, is that both induction and restriction are exact functors, and they send injective and projective Mackey functors (that is, injective and projective objects in the category $\mathrm{Mack}_R(G)$) to objects of the same type.

The notions of projectivity and injectivity are, however, distinct from those of relative projectivity and relative injectivity, which we now define. By applying M_* to the natural map of G-sets $\Omega \downarrow_H^G \uparrow_H^G \to \Omega$ we obtain a morphism of Mackey functors $M \downarrow_H^G \uparrow_H^G \to M$ specified by $M \downarrow_H^G \uparrow_H^G (\Omega) = M(\Omega \downarrow_H^G \uparrow_H^G) \to M(\Omega)$. If $\mathcal{X}$ is a set of subgroups of G we may form the morphism

$$\bigoplus_{H \in \mathcal{X}} M \downarrow_H^G \uparrow_H^G \to M$$

We define M to be *$\mathcal{X}$-projective,* or *projective relative to* $\mathcal{X}$ if and only if this morphism is a split epimorphism (in the category of Mackey functors). What this means is that for each subgroup J of G the sum of induction maps

$$\bigoplus_{H\in\mathcal{X}}\bigoplus_{x\in[J\backslash G/H]} M(J\cap {}^xH) \stackrel{(I^J_{J\cap {}^xH})}{\longrightarrow} M(J)$$

is surjective, and furthermore each of these surjections can be split in a manner compatible with inductions restrictions and conjugations. It is possible to write out these compatibility conditions explicitly, but not entirely illuminating. It is usually better to work with the abstract formalism.

Dually, we may apply M^* instead of M_* as above to obtain a morphism

$$M \to \bigoplus_{H\in\mathcal{X}} M\downarrow^G_H\uparrow^G_H .$$

We say that M is *$\mathcal{X}$-injective* if and only if this morphism is a split monomorphism.

It turns out to be convenient to express induction in a notationally different form, using G-sets. If X is a G-set and M is a Mackey functor we define a new Mackey functor M_X by $M_X(\Omega) = M(\Omega\times X)$ on objects and on morphisms as follows: if $\alpha : \Omega_1 \to \Omega_2$ then $M_{X*}(\alpha) = M_*(\alpha\times 1)$, $M^*_X(\alpha) = M^*(\alpha\times 1)$. The point about this is that in the special case when X is the G-set G/H we have $\Omega\times G/H \cong \Omega\downarrow^G_H\uparrow^G_H$ from which it follows that $M_{G/H} \cong M\downarrow^G_H\uparrow^G_H$. We define natural transformations

$$\theta_X : M_X \to M \quad \theta^X : M \to M_X$$

by putting

$$(\theta_X)_\Omega = M_*(\mathrm{pr}) : M_X(\Omega) = M(\Omega\times X) \to M(\Omega)$$
$$(\theta^X)_\Omega = M^*(\mathrm{pr}) : M(\Omega) \to M(\Omega\times X) = M_X(\Omega)$$

where $\mathrm{pr} : \Omega\times X \to \Omega$ is projection onto the first coordinate. There are some details to check to see that θ_X and θ^X are indeed natural transformations.

(3.1) PROPOSITION. *Let $\mathcal{X}$ be a set of subgroups of G and let $X = \bigsqcup_{H\in\mathcal{X}} G/H$ be the disjoint union of the transitive G-sets G/H. The following are equivalent.*
(i) M is $\mathcal{X}$-projective.
(ii) M is $\mathcal{X}$-injective.
(iii) θ_X is split surjective.
(iv) θ^X is split injective.
(v) M is a direct summand of M_X.

In view of this we may take any of these equivalent conditions as the definition of $\mathcal{X}$-projectivity. There is also an equivalent definition analogous to Higman's criterion [51].

For each Mackey functor M it may be quite important to know whether M is projective relative to some proper set of subgroups, and we need techniques to determine whether or not this is so. The following straightforward result is useful in this connection.

(3.2) LEMMA. *Let $\mathcal{X} \subseteq \mathcal{Y}$ be sets of subgroups of G.*

(i) If M is $\mathcal{X}$-projective then M is $\mathcal{Y}$-projective.

(ii) If M is $\mathcal{X}$-projective then M is $\mathcal{X}_{\max}$-projective, where $\mathcal{X}_{\max}$ is a set of representatives up to conjugacy of the maximal elements of M.

In view of this, for each set of subgroups $\mathcal{X}$, M is $\mathcal{X}$-projective if and only if M is projective relative to the closure of $\mathcal{X}$ under taking subgroups and conjugates. Provided we do this, there is in fact a unique minimal set of subgroups relative to which M is projective. This may be deduced from the next result.

(3.3) PROPOSITION.

(i) Let X and Y be G-sets and M a Mackey functor. If M is X-projective and also Y-projective then M is $X \times Y$-projective.

(ii) Let $\mathcal{X}$ and $\mathcal{Y}$ be sets of subgroups closed under taking subgroups and conjugation. If M is $\mathcal{X}$-projective and also $\mathcal{Y}$-projective then M is $\mathcal{X} \cap \mathcal{Y}$-projective.

Proof. We leave (i) as an exercise. Then (ii) follows from (i) because when we put $X = \bigsqcup_{H \in \mathcal{X}} G/H$ and $Y = \bigsqcup_{H \in \mathcal{Y}} G/H$ the stabilizers of G acting on $X \times Y$ are the groups in $\mathcal{X} \cap \mathcal{Y}$. □

We see from this that there is a unique minimal set of subgroups closed under conjugation and taking subgroups relative to which M is projective. This set is called a *defect set* (or *defect base*) for M. Thus a defect set of M is (informally) a set of subgroups $\mathcal{X}$ minimal such that the sum of the maps I from subgroups in $\mathcal{X}$ is surjective and split, in the sense previously discussed. This implies in particular that $M(G) = \sum_{H \in \mathcal{X}} I_H^G M(H)$ and it was this condition alone which Green used in his definition of a defect set. However, he was working in the context of Green functors, and in that case this condition is sufficient to imply everything else as we are about to see.

(3.4) THEOREM (Dress [25, Theorem 1]). *Let M be a Green functor and $\mathcal{X}$ a set of subgroups of G. For M to be $\mathcal{X}$-projective (as a Mackey functor) it suffices that the sum of the induction maps*

$$(I_H^G) : \bigoplus_{H \in \mathcal{X}} M(H) \to M(G)$$

be surjective.

This makes it easy to deduce the relative projectivity of many familiar examples of Mackey functors, and in many cases to find their defect sets. So, for example, the character rings $\mathbb{Q} \otimes_{\mathbb{Z}} G_0(kG)$ where k is a field have as their defect sets all cyclic subgroups of G in case $\operatorname{char} k = 0$, and all cyclic p'-subgroups in case $\operatorname{char} k = p$. In the characteristic zero case, Artin's induction theorem coupled with Theorem 3.4 gives projectivity relative to cyclic subgroups. No smaller set of subgroups is possible since for no cyclic group is the sum of the induction maps from proper subgroups surjective. For a full discussion of these defect sets, as well as those of $G_0(kG)$ and the Green ring A(G) see sections 9 and 10 of [56].

Provided $n \geq 1$ the cohomology Mackey functor $H^n(G,U)$ is the direct sum of functors giving the p-torsion subgroup $H^n(G,U)_p$. Each p-torsion functor is projective relative to p-subgroups of G. We may see this either using Theorem 3.4 applied to $H^*(G,\mathbb{Z})_p$ since the corestriction from a Sylow p-subgroup is surjective, and then quoting further theory to do with the fact that cohomology in general is a Green module over $H^*(G,\mathbb{Z})$; or for a different approach, see Section 7. We deduce that $H^n(G,U)$ is projective relative to the set of all p-subgroups for all the prime divisors of $|G|$.

In general $H^n(G,U)_p$ may have a defect set smaller than all p-subgroups, depending on the module U. For example, if $K \leq G$ then $H^0(\_\,, V\uparrow_K^G) \cong (H^0(\_\,, V))\uparrow_K^G$ as Mackey functors [62, 5.2], and so this functor is projective relative to K. On the other hand $H^0(\_\,,\mathbb{Z}/p\mathbb{Z})$ has defect set all p-subgroups since if H is a p-subgroup which is not a Sylow p-subgroup then the corestriction map $I_H^G = 0$. From this we may see by dimension shifting that for each n there is a choice of module U so that $H^n(\_\,,U)_p$ has defect set all p-subgroups of G.

The Burnside ring Mackey functor $B(G)$ has defect set all subgroups of G since the G-set consisting of a single point is never an orbit in a properly induced G-set.

Dress observed [25] that under the hypothesis of relative projectivity, not only is the value of a Mackey functor the sum of the images of induction maps, but that also the kernel of this map is determined. To show this he studied a resolution of the Mackey functor which he called an *Amitsur complex*, and which we now describe.

We suppose that X is a finite G-set and let $X^r = X \times \cdots \times X$ denote the r-fold product of X with itself. Let $\mathrm{pr}_{\bar{i}} : X^r \to X^{r-1}$ denote projection off component i. We consider the complex of Mackey functors

$$C: \quad \cdots \xrightarrow{d_2} M_{X^2} \xrightarrow{d_1} M_X \xrightarrow{d_0} M \longrightarrow 0$$

which evaluated on Ω is

$$\begin{array}{ccccccccccc} & & & M(\Omega\times X\times X) & & M(\Omega\times X) & & & & \\ & & & \| & & \| & & & & \\ C(\Omega): & \cdots & \xrightarrow{d_2} & M_{X\times X}(\Omega) & \xrightarrow{d_1} & M_X(\Omega) & \xrightarrow{d_0} & M(\Omega) & \longrightarrow & 0 \end{array}$$

where

$$d_r = \sum_{i=0}^{r} (-1)^i M_*(1 \times \mathrm{pr}_{\bar{i}}).$$

Thus $d_0 = \theta_X$, and by a standard calculation we verify that $d_r d_{r-1} = 0$.

There is a similar construction using M^* which gives a complex

$$D: \quad 0 \longrightarrow M \xrightarrow{d_0} M_X \xrightarrow{d_1} M_{X^2} \xrightarrow{d_2} \cdots$$

with $d_0 = \theta^X$. The next result implies that the complexes C and D are acyclic in case M is X-projective or, equivalently, X-injective.

(3.5) THEOREM. *If M is X-projective then both C and D are chain homotopic to the zero complex.*

Proof. It is useful to say that a chain complex is *contractible* if it is chain homotopic to the zero complex. Summands of contractible complexes are contractible. Since M is a summand of M_X it suffices to prove the result for the functor M_X. The complex we obtain replacing M by M_X evaluated at a G-set Ω is

$$\begin{array}{ccccccc} & & M(\Omega\times X^{r+1}\times X) & & M(\Omega\times X^{r}\times X) & & \\ & & \| & & \| & & \\ \cdots & \xrightarrow{d_{r+1}} & M_X(\Omega\times X^{r+1}) & \xrightarrow{d_r} & M_X(\Omega\times X^{r}) & \xrightarrow{d_{r-1}} & \cdots. \end{array}$$

It is a routine check that the degree 1 mapping $s_r : M(\Omega\times X^r\times X)\to M(\Omega\times X^{r+1}\times X)$ given by $s_r = (-1)^{r-1}M_*(1\times 1^r\times\Delta)$, where $\Delta : X\to X\times X$ is the diagonal, satisfies $s_{r-1}d_r + d_{r+1}s_r = 1$, showing that the identity mapping on this complex is chain homotopic to 0. □

To say that the complexes C and D are contractible is equivalent to saying that they are isomorphic to a direct sum of complexes of the form $\cdots\to 0\to A\xrightarrow{\alpha}A\to 0\to\cdots$ where α is an isomorphism. This means that the complexes are acyclic, and are everywhere split. The acyclicity implies that the values $M(K)$ of the Mackey functor are given as the cokernel (if we use M_*) or the kernel (if we use M^*) of the explicitly given map d_1, and this description is compatible with inclusions of subgroups and conjugations in G.

The cokernel of the map $M_{X^2}(G)\to M_X(G)$ in the complex C may be described as a colimit, and the kernel of $M_X(G)\to M_{X^2}(G)$ in complex D may be described as a limit, as we may see in an elementary fashion. We will see in the next section that the other homology groups of C and D can be interpreted as derived functors of colimit and limit functors. For these results we only really need to work with half of the Mackey functor, either the covariant half M_* — which is known as a *coefficient system* — or the contravariant half M^*.

(3.6) PROPOSITION.

(i) In the complex C, $\mathrm{Coker}(M_{X^2}(G)\xrightarrow{d_1}M_X(G))$ *is the colimit of the diagram made up of all possible morphisms of R-modules*

$$M(H^g\cap K)\xrightarrow{I^K_{H^g\cap K}}M(K)\qquad\text{and}\qquad M(H^g\cap K)\xrightarrow{I^H_{H\cap {}^gK}c_g}M(H)$$

where $H,K\in\mathcal{X}$ and $g\in G$.

(ii) In the complex D, $\mathrm{Ker}(M_X(G)\xrightarrow{d_1}M_{X^2}(G))$ *is the limit of the diagram made up of all possible morphisms of R-modules*

$$M(K)\xrightarrow{R^K_{H^g\cap K}}M(H^g\cap K)\qquad\text{and}\qquad M(H)\xrightarrow{c_{g^{-1}}R^H_{H\cap {}^gK}}M(H^g\cap K)$$

where $H, K \in \mathcal{X}$ and $g \in G$.

Proof. In a similar way to Lemma 2.1 (note that $X \times X$ is the pullback of $X \to \mathrm{pt} \leftarrow X$) we see in the covariant case that $M_{X^2} \to M_X$ is the direct sum of terms with component maps $M_{G/H^g \cap K} \to M_{G/H} \oplus M_{G/K}$ specified by $(-I^H_{H\cap {}^gK} c_g, I^K_{H^g\cap K})$. The cokernel of this is the stated colimit, and the contravariant case is similar. □

This observation provides the connection with one of the main examples which motivates this general development, namely the 'stable elements' formula of Cartan and Eilenberg [16]. If we take M to be the p-part of group cohomology and $\mathcal{X}$ to be all p-subgroups of G then the assertion that $H^n(G, U)_p$ is isomorphic to the stable elements in $H^n(P, U)$, where P is a Sylow p-subgroup is exactly the assertion that it is isomorphic to the limit of the above-mentioned diagram. We thus see how to generalize this formula to arbitrary Mackey functors.

(3.7) COROLLARY. *If M is $\mathcal{X}$-projective then $M(G) \cong \mathrm{colim}_{\mathcal{X}} M_* \cong \lim_{\mathcal{X}} M^*$ where these terms denote the colimit and limit described in the last result. In particular, if M is a Green functor and the sum of the induction maps from subgroups in $\mathcal{X}$ to $M(G)$ is surjective, then these isomorphisms hold.*

4. Complexes obtained from G-spaces

A deficiency of the Amitsur complex considered by Dress is that it has infinite length, and it is often more useful to have a resolution of finite length. The Amitsur complex is in fact a particular case of a theory in which we obtain exact sequences of Mackey functors from the action of G on a suitable space and we now describe this. We first have to say with what kind of G-spaces we will work, and the most elegant formulation is to define a *G-space* to be a simplicial G-set, that is, a simplicial object in the category of G-sets. For the reader unfamiliar with these we may equally consider admissible G-CW complexes (or admissible G-simplicial complexes), namely CW complexes (simplicial complexes) equipped with a cellular (simplicial) action of G and satisfying the condition that for each cell (simplex) σ the stabilizer G_σ fixes σ pointwise. If Z is a G-space we denote the set of (non-degenerate) simplices (or cells) in dimension i by Z_i. For each i this is a G-set.

Given a Mackey functor M we may construct a covariant functor $F_M : G\text{-set} \to \mathrm{Mack}_R(G)$ defined by $F_M(X) = M_X$, using the covariant part of the functor M_* to give the functorial dependence on X. We may also define a contravariant functor $F^M : G\text{-set}^{\mathrm{op}} \to \mathrm{Mack}_R(G)$ defined again by $F^M(X) = M_X$, but using the contravariant part of the functor M^* to give the functorial dependence on X. Now given a G-space Z we may construct complexes of Mackey functors

$$\cdots \to M_{Z_1} \to M_{Z_0} \to M \to 0$$

and

$$\cdots \leftarrow M_{Z_1} \leftarrow M_{Z_0} \leftarrow M \leftarrow 0$$

as follows. Regarding Z as a functor $Z : \Delta^{\rm op} \to$G-set (where Δ is the category of sets of the form $\{0, \ldots, n\}$ with monotone maps as morphisms) we obtain by composition a simplicial Mackey functor $F_M \circ Z$. The first sequence is now the normalized chain complex of this simplicial object, augmented by the map $\theta_{Z_0} : M_{Z_0} \to M$. The second sequence is obtained similarly from $F^M \circ Z^{\rm op}$, augmenting by the map $\theta^{Z_0} : M \to M_{Z_0}$. For future reference, let us denote by M_Z the sequence $\cdots \to M_{Z_1} \to M_{Z_0} \to 0$ without augmentation, and by M^Z the sequence $\cdots \leftarrow M_{Z_1} \leftarrow M_{Z_0} \leftarrow 0$ again without augmentation.

As an example we indicate how the Amitsur complex which Dress considered may be constructed in this way. Given a G-set X, we construct a space Z as the nerve of the category in which the objects are the elements of X, and in which there is precisely one morphism (an isomorphism) between each ordered pair of objects. Thus the r-simplices (including the degenerate ones) are in bijection with X^{r+1}. Dress's complex is the (unnormalized) chain complex of $F_M \circ Z$, augmented by θ_{Z_0}. The augmented normalized complex is a quotient of this by a contractible subcomplex, and has the same homology.

The following is the theorem which ties all this together.

(4.1) THEOREM ([65], [9], [27]). *Let G be a finite group, M a Mackey functor for G, $\mathcal{X}$ and $\mathcal{Y}$ sets of subgroups of G which are closed under taking subgroups and conjugation, and Z a G-space. Suppose that*

(i) M is projective relative to $\mathcal{X}$.

(ii) For every $Y \in \mathcal{Y}$, $M(Y) = 0$.

(iii) For every subgroup $H \in \mathcal{X} - \mathcal{Y}$, the fixed points Z^H are contractible.

Then the complexes of Mackey functors

$$\cdots \to M_{Z_1} \to M_{Z_0} \to M \to 0 \qquad \text{and} \qquad \cdots \leftarrow M_{Z_1} \leftarrow M_{Z_0} \leftarrow M \leftarrow 0$$

are contractible; that is, they are acyclic and everywhere split.

In the case of the Amitsur complex we take $X = \bigsqcup_{H \in \mathcal{X}} G/H$ and $\mathcal{Y} = \emptyset$. Then for every $H \in \mathcal{X}$ the space Z previously constructed satisfies the condition that Z^H is contractible, since it is the nerve of the category whose objects are the elements of X^H and where there is a single morphism between each pair of objects. This category is equivalent to a category with only one object and morphism, and its nerve is contractible.

When we evaluate the sequences of the theorem at G we get sequences which express $M(G)$ in terms of the values of M on the stabilizer groups of the simplices in Z. Thus the covariant sequence may be written

$$\cdots \to \bigoplus_{\sigma \in [G \backslash Z_1]} M(G_\sigma) \to \bigoplus_{\sigma \in [G \backslash Z_0]} M(G_\sigma) \to M(G) \to 0.$$

This is particularly useful when the G-space Z has finite dimension, in which case the sequences have finite length and the acyclicity and splitting mean that the isomorphism type of $M(G)$ is determined by the isomorphism types of the remaining terms.

This approach has been used quite extensively to assist in the computation of group cohomology, and a description of these applications is given in [1] (in a more rudimentary version phrased only in terms of group cohomology, and without the force of the exact sequence). For this we fix a prime p and let $M(G) = H^n(G,U)_p$ be the Sylow p-subgroup of the group cohomology in degree n of G with coefficients in the $\mathbb{Z}G$-module U, for some $n > 0$ and U. For Z we may take various spaces, for example the order complex (i.e. the nerve) of the poset

$$\mathcal{S}_p(G) = \{H \leq G \bigm| 1 \neq H \text{ is a } p\text{-subgroup}\}$$

with G acting by conjugating the subgroups, or equally one of a number of other G-spaces (see [27], [64]). We take $\mathcal{X}$ to be all p-subgroups of G and $\mathcal{Y}$ to contain just the identity subgroup. Then the conditions of the theorem are satisfied, and the isomorphism type of $H^n(G,U)_p$ is conveniently expressed by the equation

$$H^n(G,U)_p = \sum_{\sigma \in [G \backslash |\mathcal{S}_p(G)|]} (-1)^{\dim \sigma} H^n(G_\sigma, U)_p$$

the sum being over representatives of the G-orbits of (non-degenerate) simplices in the order complex $|\mathcal{S}_p(G)|$. This equation holds in the Grothendieck group of finite abelian groups with relations given by direct sum decompositions.

We should mention also that in this context the truncated sequences $M_Z(G)$ and $M^Z(G)$ make up the E^1 and E_1 pages of what we may call (c.f. [27]) the 'isotropy spectral sequences' for the equivariant homology and cohomology of G acting on Z.

There is another interpretation of the sequences M_Z and M^Z, which is that they compute the derived functors of certain limit and colimit functors. These have importance because they appear in the spectral sequence of Bousfield and Kan (see [27]). It is also interesting to have an interpretation like this of the homology of complexes such as Dress's Amitsur complex. The general framework is that we have a small category $\mathcal{C}$ and another category $\mathcal{D}$. Let $\mathcal{D}^{\mathcal{C}}$ denote the category of functors from $\mathcal{C}$ to $\mathcal{D}$, and let $\lim, \operatorname{colim} : \mathcal{D}^{\mathcal{C}} \to \mathcal{D}$ denote the limit and colimit functors (assuming they exist). If $\mathcal{D}$ is an abelian category then so is $\mathcal{D}^{\mathcal{C}}$, and we may consider the right derived functors $\lim^i$ of lim and the left derived functors colim_i of colim.

We need to consider the situation where our G-space Z is constructed as the homotopy colimit $Z = \operatorname{hocolim} \alpha$ of a diagram $\alpha : \mathcal{C} \to G$-set. For this construction see [27], where it is shown that $|\mathcal{S}_p(G)|$ and many other spaces may be constructed in this way up to equivariant homeomorphism.

As an example we show how to construct the space which gives the Amitsur complex in this way. Starting with a G-set X we let $\mathcal{C}$ be the category whose objects are the orbits of G on the various sets X^{r+1}, $r \geq 0$ and where the morphisms $\Omega \to \Psi$ are the restrictions of all possible projection mappings $\mathrm{pr}_I : X^{r+1} \to X^{|I|}$ where I is a subset of the set $\{0, 1, \ldots, r\}$ indexing the product X^{r+1}. Let $\alpha : \mathcal{C} \to G$-set be the inclusion functor. Now $\mathrm{hocolim}\,\alpha$ is G-homeomorphic to the space described earlier which gives Dress's Amitsur complex. This can be proved using the methods of [27], by showing that the 'Grothendieck construction' of α is a category whose nerve is the desired space.

Given a Mackey functor M and a diagram of G-sets $\alpha : \mathcal{C} \to G$-set, we obtain diagrams of R-modules by composition $M_* \circ \alpha : \mathcal{C} \to R$-mod and $M^* \circ \alpha^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to R$-mod.

(4.2) PROPOSITION. *Let $Z = \mathrm{hocolim}\,\alpha$ where $\alpha : \mathcal{C} \to G$-set. Then*

$$\mathrm{colim}_i(M_* \circ \alpha) \cong H_i(M_Z),$$

and

$$\lim{}^i(M^* \circ \alpha) \cong H_i(M^Z).$$

Proof. The idea of the proof is that the construction of the homotopy colimit of the diagram α of sets may also be done with the diagram $M_* \circ \alpha$ of R-modules, in which case the result is a simplicial R-module whose degree i homology is $\mathrm{colim}_i(M_* \circ \alpha)$ [28, App. II, 3.3]. We get the same answer if we form $\mathrm{hocolim}\,\alpha$, apply M_* and take homology. The argument for the second isomorphism involving $\lim^i$ is dual. □

As a consequence of this identification and the previous theorem we obtain the following corollary.

(4.3) COROLLARY. *Let M be a Mackey functor for G and $\mathcal{X}$ and $\mathcal{Y}$ be sets of subgroups of G which are closed under taking subgroups and conjugation. Suppose that*

(i) M is projective relative to $\mathcal{X}$.

(ii) For every $Y \in \mathcal{Y}$, $M(Y) = 0$.

Let $\mathcal{C}$ be the full subcategory of G-set whose objects are the coset spaces G/H where $H \in \mathcal{X} - \mathcal{Y}$. Then $M_* : \mathcal{C} \to R$-mod and $M^* : \mathcal{C}^{\mathrm{op}} \to R$-mod may be regarded as diagrams of R-modules and we have $\lim^i M^* = 0$ and $\mathrm{colim}_i M_* = 0$ for all $i > 0$.

Proof. Let $\alpha : \mathcal{C} \to G$-set be the inclusion functor. The space $\mathrm{hocolim}\,\alpha$ is considered in [27] where it is denoted $X_{\mathcal{C}}^{\beta}$, and it is shown that if $H \in \mathcal{X} - \mathcal{Y}$ then $(\mathrm{hocolim}\,\alpha)^H$ is contractible. Thus the conditions of the previous theorem are satisfied and we deduce that $H_i(M_{\mathrm{hocolim}\,\alpha}) = 0 = H_i(M^{\mathrm{hocolim}\,\alpha})$ when $i > 0$. □

5. Mackey functors as representations of the Mackey algebra

The structure of Mackey functors may be analyzed in a similar way to the representation theory of finite groups with many similarities in the results. We describe this approach in this section.

To start with, Mackey functors really are the same thing as modules for a certain finite-dimensional algebra defined by Thévenaz and Webb in [62] as follows. As always we work over a commutative ground ring R, which has a 1. We consider the free algebra on non-commuting variables $I_H^K, R_H^K, c_{x,H}$ where H and K range over subgroups of G with $H \leq K$, and x ranges over elements of G. The *Mackey algebra* $\mu_R(G)$ is the quotient of this algebra by the ideal given by the following relations.

(0) $I_H^H = R_H^H = c_{h,H}$ for all subgroups H and $h \in H$

(1) $R_J^K R_K^H = R_J^H$
(2) $I_K^H I_J^K = I_J^H$ } for all subgroups $J \leq K \leq H$

(3) $c_{g,{}^hK} c_{h,K} = c_{gh,H}$ for all $g, h \in G$ and subgroups K

(4) $R_{{}^gK}^{{}^gH} c_{g,H} = c_{g,K} R_K^H$
(5) $I_{{}^gK}^{{}^gH} c_{g,K} = c_{g,H} I_K^H$ } for all subgroups $K \leq H$ and $g \in G$

(6) $R_J^H I_K^H = \sum_{x \in [J \backslash H/K]} I_{J \cap {}^xK}^J c_x R_{J^x \cap K}^K$ for all subgroups $J, K \leq H$

(7) $\sum_{H \leq G} I_H^H = 1$

(8) All other products of $I_H^L, R_J^K, c_{h,Q}$ are zero.

A Mackey functor M may be regarded as the $\mu_R(G)$-module $\bigoplus_{H \leq G} M(H)$ where the generators I_H^K, R_H^K and $c_{x,H}$ act on each summand in this direct sum as the mappings $I_H^K : M(H) \to M(K)$, $R_H^K : M(K) \to M(H)$, $c_{x,H} : M(H) \to M({}^xH)$ where this is possible, and as zero on other summands.

It is immediate to see that the ideal of relations will act as zero, since these relations are part of the definition of the Mackey functor. Conversely we note that $1 = \sum_{H \leq G} I_H^H$ is a sum of orthogonal idempotents, and so if we have a $\mu_R(G)$-module V we may write it as $V = \bigoplus_{H \leq G} I_H^H \cdot V$. The specification $M(H) = I_H^H \cdot V$ defines a Mackey functor, with the action of I_H^K, R_H^K and $c_{x,H}$ coming from the module structure.

The standard approaches to the representation theory of finite-dimensional algebras may now be applied to Mackey functors. This has been done in [9], [37], [51], [60], [62], [67] to name just a few sources, and there is a summary in [57]. We will describe some of the principal ideas. We will refer to Mackey functors, but we could equally refer to $\mu_R(G)$-modules, and similarly we will refer to subfunctors instead of submodules.

There are simple Mackey functors (having no proper subfunctors), and if R is a field (or a complete local ring) they have projective covers, which form a complete list of indecomposable projective Mackey functors. These simple Mackey functors are parametrized and explicitly described in [60], where it is also proved that if R is a field of characteristic 0 or of characteristic not dividing $|G|$ then $\mu_R(G)$ is semisimple (see also [62, (14.4)]). There is a decomposition map analogous to that for group representations. It is surjective, the Cartan matrix satisfies the equation $C = D^t D$ (where D is the decomposition matrix) and hence it is symmetric and non-singular [62]. This also provides a very effective way to compute the Cartan matrix. Further information about the projectives of a rather deep and fundamental nature is given in [9].

The notion of relative projectivity was developed in [51] into a theory of vertices and sources, as well as Green correspondence. We have seen that for each Mackey functor M there is a unique set of subgroups $\mathcal{X}$ closed under conjugation and taking subgroups, minimal with respect to the property that M is $\mathcal{X}$-projective. If R is a field or a complete discrete valuation ring and M is indecomposable this set consists of a single conjugacy class of subgroups together with their subgroups. A representative of this single conjugacy class is called a *vertex* of M. The notions of source and Green correspondence are now formulated in the usual way. Unlike the situation with group representations in characteristic p, the vertex of an indecomposable Mackey functor need not be a p-subgroup of G; in fact any subgroup of G may be the vertex of an indecomposable Mackey functor, even when the Mackey functor is projective. This points to another difference with group representations, which is that whereas an indecomposable Mackey functor whose vertex is the identity and whose values are projective R-modules is necessarily projective, the converse is not true (assuming $G \neq 1$).

Various techniques are available to analyze in detail the subfunctor structure of a specific Mackey functor. A method is described in [62] to find the composition factors of the Mackey functor, and there is developed a way to compute Ext groups between the simple functors. We generally expect the subfunctor structure of a Mackey functor to be more complicated than the submodule structure of representations of the same groups, but still in small cases it can be done. In [62] it is proved that when R is a field of characteristic p and p divides $|G|$ to the first power, but not the second, $\mu_R(G)$ is a direct sum of semisimple algebras and Brauer tree algebras in an explicitly given way, so that all Mackey functors can be completely described in this situation. Such algebras are self-injective and of finite representation type. It is proved that if $p^2 \mid |G|$ then $\mu_R(G)$ is neither self-injective, nor of finite representation type.

The Mackey algebra is a direct sum of indecomposable ideal summands, and these are the *blocks* of Mackey functors. In [62] these are explicitly parametrized in terms of the blocks of G and its sections, and properties are described which enable us to determine the block to which a given indecomposable Mackey functor belongs.

6. Induction theorems and the action of the Burnside ring

The Burnside ring plays a particularly important role with regard to Mackey functors. On the one hand it provides an example of a Mackey functor $M(G) = B(G)$, which is in fact a Green functor. As a Mackey functor, B is generated by the (isomorphism class of the) G-set which consists of a single point. Furthermore it satisfies a universal property, that given any Mackey functor N, every assignment $\eta(\text{point}) \in N(G)$ extends uniquely to a morphism of Mackey functors $\eta : B \to N$. It follows from this that B is a projective object in $\text{Mack}_R(G)$. More generally, if we denote by B^H the Burnside ring functor as a Mackey functor on H and its subgroups, then $B^H \uparrow_H^G$ is a projective Mackey functor (since induction carries projectives to projectives). If we assume R is a field or a complete discrete valuation ring then every indecomposable projective Mackey functor is a summand of some $B^H \uparrow_H^G$ (see [62, 8.6]).

Turning to another structure, there is an action of the ring $B(G)$ as a ring of endomorphisms of every Mackey functor for G. This action may defined in several equivalent ways. In terms of G-set notation, if X is a finite G-set and M a Mackey functor we have previously defined (in the context of relative projectivity) natural transformations

$$M \xrightarrow{\theta^X} M_X \xrightarrow{\theta_X} M.$$

We define X to act on M as the composite $\theta_X \theta^X$. It is hard to see at first what this composite is doing. In the particular case when $X = G/K$ for some subgroup K, the effect on $M(G/H)$ is a composite of maps $M(G/H) \to M(G/H \times G/K) \to M(G/H)$ where we have an identification

$$M(G/H \times G/K) \cong \bigoplus_{g \in [H \backslash G/K]} M(G/H \cap {}^gK).$$

From this we may see that if $x \in M(G/H)$ then

$$G/K \cdot x = \sum_{g \in [H \backslash G/K]} I^H_{H \cap {}^gK} R^H_{H \cap {}^gK}(x).$$

Yet another way to specify the action of the Burnside ring is to observe that there is an R-algebra homomorphism $B(G) \to \mu_R(G)$ specified on basis elements by

$$G/K \mapsto \sum_{H \leq G} \sum_{g \in [H \backslash G/K]} I^H_{H \cap {}^gK} R^H_{H \cap {}^gK}.$$

This homomorphism is injective [62], and because the resulting action commutes with the Mackey functor operations, it embeds $B(G)$ as a subalgebra of the center of $\mu_R(G)$.

It follows from this that any expression $1 = e_1 + \cdots + e_r$ in $B(G)$ as a sum of orthogonal idempotents gives a decomposition of every Mackey functor as $M = e_1 M \oplus \cdots \oplus e_r M$, and that the indecomposable summands of each e_iM lie in distinct blocks from the summands of the other e_jM with $j \neq i$. This is because blocks may be identified as the primitive central idempotents in $\mu_R(G)$, and each e_i is a sum of blocks.

It becomes important to have a description of the primitive idempotents in $B(G)$. When $|G|$ is invertible in R, $B(G)$ is semisimple and an explicit description of the idempotents appears in [68] and [29]. In practical applications with Mackey functors whose values have torsion it is helpful to know the result of Dress [23] which shows that when p is a prime and all prime divisors of $|G|$ other than p are invertible in R, the primitive idempotents in $B(G)$ are in bijection with conjugacy classes of p-perfect subgroups of G. (A subgroup J is p-perfect if it has no non-identity p-group as a homomorphic image.) Writing f_J for the corresponding primitive idempotent of $B(G)$, several descriptions are given in [62] which characterize the summand f_JM of M. In particular when R is additionally a field or complete discrete valuation ring, the summands of f_JM are precisely the summands of M which have a vertex containing J as a normal subgroup of p-power index. This means we can tell which of the summands f_JM are non-zero by knowing a defect set of M. For example, if $M(G) = H^n(G,U)_p$ is the Mackey functor given by taking the p-torsion subgroup of group cohomology in degree n with some $\mathbb{Z}G$-module U, a defect set will consist entirely of p-subgroups of G. Here we may regard M as a Mackey functor over $R = \mathbb{Z}_p$, the p-adic integers. Every indecomposable summand of M has a p-subgroup as a vertex. Since the only p-perfect subgroup of a p-group is the identity subgroup, we have $f_1M = M$, all other summands being zero, and so the decomposition of M given by Burnside ring idempotents is of no help in examining the structure of M. On the other hand, when the defect set of M is larger there may be more summands and useful information may be obtained. This is exemplified very nicely in [49] with the algebraic K-theory of $\mathbb{Z}G$.

We now describe an induction theorem of Dress, and for this we introduce certain subfunctors of a Mackey functor M. Let $\mathcal{X}$ be a set of subgroups of G which is closed under conjugation and taking subgroups and put $X = \bigsqcup_{H\in\mathcal{X}} G/H$. We will again use the natural transformations $\theta_X : M_X \to M$ and $\theta^X : M \to M_X$ from Section 3, and write $I_{\mathcal{X}}M = \theta_X(M)$ and $R_{\mathcal{X}}M = \operatorname{Ker}\theta^X$. These are subfunctors of M and the letters in the notation are suggested by the fact that they are specified as the image of induction maps from subgroups in $\mathcal{X}$, and the kernel of restriction maps to subgroups in $\mathcal{X}$, respectively.

(6.1) LEMMA. *Let $\mathcal{X}$ be a set of subgroups of G closed under conjugation and taking subgroups. Then*

$$I_{\mathcal{X}}M(K) = \sum_{J\leq K,\, J\in\mathcal{X}} I_J^K M(J)$$

and

$$R_{\mathcal{X}}M(K) = \bigcap_{J\leq K,\, J\in\mathcal{X}} \operatorname{Ker} R_J^K .$$

If π is a set of primes (possible empty) we write π' for the complementary set of primes and let $|G| = |G|_\pi \cdot |G|_{\pi'}$ be the product of numbers whose prime divisors lie respectively in π and π'. Given a set of subgroups $\mathcal{X}$ we will write

$$\mathcal{H}_\pi\mathcal{X} = \{K \leq G \bigm| \text{there exist } p \in \pi \text{ and } H \triangleleft K \text{ with } H \in \mathcal{X} \text{ and } K/H \text{ a } p\text{-group}\}.$$

(6.2) THEOREM (Dress [25, Theorems 2 and 4], [24, Theorem 7.1]). *Let $\mathcal{X}$ be a set of subgroups of G closed under conjugation and taking subgroups and let π be a set of primes. We have*

$$|G|_{\pi'}M \subseteq I_{\mathcal{H}_\pi\mathcal{X}}M + R_{\mathcal{X}}M$$

and

$$|G|_{\pi'} \cdot (I_{\mathcal{X}}M \cap R_{\mathcal{H}_\pi\mathcal{X}}M) = 0.$$

In his original formulation Dress stated this result only for the evaluation of the Mackey functors at G. In view of the identifications given in the preceding lemma, the form of the result we have given is immediate.

This theorem is perhaps most useful when we take π either to be empty, or to be all primes. Evidently $\mathcal{H}_\emptyset\mathcal{X} = \mathcal{X}$. When π consists of all primes let us simply write $\mathcal{H}\mathcal{X}$ instead of $\mathcal{H}_\pi\mathcal{X}$. In these cases we obtain:

(6.3) COROLLARY. *For any set of subgroups $\mathcal{X}$ closed under conjugation and taking subgroups we have*

$$|G| \cdot M \subseteq I_{\mathcal{X}}M + R_{\mathcal{X}}M, \quad |G| \cdot (I_{\mathcal{X}}M \cap R_{\mathcal{X}}M) = 0$$

and

$$M = I_{\mathcal{H}\mathcal{X}}M + R_{\mathcal{X}}M, \quad I_{\mathcal{X}}M \cap R_{\mathcal{H}\mathcal{X}}M = 0.$$

As a consequence:

(6.4) COROLLARY. *Suppose that $|G|$ is invertible in R. Then*

$$M = I_{\mathcal{X}}M \oplus R_{\mathcal{X}}M.$$

An example of the application of this is Conlon's theorem [19, (81.36)] giving a decomposition of the Green ring $A(G)$ of finitely generated kG-modules where k is either a field of characteristic p or a complete discrete valuation ring with residue field of characteristic p. If $P \leq G$ is a p-subgroup we consider the subspace U of $A(G)$ spanned by the modules which are relatively P-projective and the subspace V spanned by all expressions $[B] - [C] - [A]$ arising from short exact sequences of kG-modules $0 \to A \to B \to C \to 0$ which split on restriction to P. Then taking

$$\mathcal{X} = \{H \bigm| H/O_p(H) \text{ is cyclic}, O_p(H) \text{ is conjugate to a subgroup of } P\}$$

it follows from the discussion in [19] that $U = I_{\mathcal{X}}A(G)$ and $V = R_{\mathcal{X}}A(G)$. We have $A(G) = U \oplus V$.

At this point we should mention that the semisimplicity of the category of Mackey functors over a field R of characteristic 0 mentioned in Section 5 allows us to say that every subfunctor of a Mackey functor M over R, and in particular $I_{\mathcal{X}}M$, is a direct summand of M. The extra information in Dress's result is that it identifies $R_{\mathcal{X}}M$ as a direct complement.

We also point out that if we have several sets of subgroups of G in a chain

$$\mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots \subset \mathcal{X}_n$$

then evidently

$$I_{\mathcal{X}_0}M \subseteq I_{\mathcal{X}_1}M \subseteq \cdots \subseteq I_{\mathcal{X}_n}M$$

and

$$R_{\mathcal{X}_0}M \supseteq R_{\mathcal{X}_1}M \supseteq \cdots \supseteq R_{\mathcal{X}_n}M.$$

When $|G|$ is invertible in R we have

$$M = I_{\mathcal{X}_i}M \oplus R_{\mathcal{X}_i}M$$

for each i and by the modular law we have

$$I_{\mathcal{X}_i}M = I_{\mathcal{X}_{i-1}}M \oplus (I_{\mathcal{X}_i}M \cap R_{\mathcal{X}_{i-1}}M).$$

Hence

$$M = \bigoplus_{i=0}^{n} I_{\mathcal{X}_i}M \cap R_{\mathcal{X}_{i-1}}M$$

where $R_{\mathcal{X}_{-1}}M = M$. This decomposition is exemplified by a different part of Conlon's theorem [19, (81.36)].

The penultimate equality in Corollary 6.3 is very useful in obtaining induction theorems in situations where we know $R_{\mathcal{X}}M$ to be zero. So for example there is an induction theorem due to Conlon [19, (80.50)] for the Green ring $A(G)$ over $\mathbb{Q}$ of finitely generated kG-modules where k is either a field of characteristic p or a complete discrete valuation ring with residue field of characteristic p. It says that if $\mathcal{X} = \{H \leq G \bigm| H/O_p(H) \text{ is cyclic}\}$ then $A(G)$ is the sum of the images of the induction maps from the subgroups in $\mathcal{X}$, or in our language $A = I_{\mathcal{X}}A$. From this and Corollary 6.4 we obtain $R_{\mathcal{X}}A = 0$. If we now let $a(G)$ denote the Green ring over $\mathbb{Z}$ of finitely generated kG-modules, allowing only linear combinations over $\mathbb{Z}$, we deduce that $R_{\mathcal{X}}a = 0$, since it embeds in $R_{\mathcal{X}}A = 0$. We deduce from Corollary 6.3 that $a = I_{\mathcal{H}\mathcal{X}}a$, which is the integral form of Conlon's induction theorem, due to Dress in [26].

7. Cohomological Mackey functors

A Mackey functor is said to be *cohomological* if for every pair of subgroups $H \leq K$ of G the map $I_H^K R_H^K : M(K) \to M(K)$ is multiplication by $|K : H|$. These functors take their name because group cohomology $M(K) = H^n(K, U)$ satisfies this condition. The functor which assigns to each subgroup of the Galois group of an extension of number fields the class group of the ring of integers of the fixed field is another example of a cohomological Mackey functor, since the Mackey functor operations derive from taking fixed points.

Perhaps the most striking result about cohomological Mackey functors is the theorem of Yoshida which identifies them as modules for the Hecke algebra

$$\mathcal{E} = \mathrm{End}_{RG}(\bigoplus_{H \leq G} R[G/H]),$$

the endomorphism ring of the direct sum of all permutation modules $R[G/H]$ where H ranges over the subgroups of G. They are also related to the category $\mathcal{H}_G$ defined to be the full subcategory of RG-modules whose objects are the finitely generated permutation RG-modules.

(7.1) THEOREM (Yoshida [69]). *The following categories are equivalent:*

(i) the full subcategory of $\mathrm{Mack}_R(G)$ *whose objects are the cohomological Mackey functors,*

(ii) the category of R-linear functors $\mathcal{H}_G \to R$-mod, and

(iii) the category of $\mathcal{E}$-modules.

This result identifies $\mathcal{E}$ with what we might call the 'cohomological Mackey algebra', obtained by imposing the relations $I_H^K R_H^K = |K : H| \cdot I_K^K$ on the Mackey algebra. The equivalence of (ii) and (iii) is a routine piece of category theory, immediate from the definitions.

What is behind the equivalence of (i) and (ii) is that all morphisms in $\mathcal{H}_G$ can be expressed as linear combinations of composites of three kinds of morphism: the morphisms

$$i_H^K : R[G/H] \to R[G/K]$$
$$r_H^K : R[G/K] \to R[G/H]$$
$$c_g : R[G/H] \to R[G/{}^gH]$$

specified by $i_H^K(xH) = xK$ and $r_H^K(xK) = \sum_{k \in [K/H]} xkH$ whenever H is a subgroup of K, and $c_g(xH) = xg^{-1}\,{}^gH$ whenever $g \in G$ and $H \leq G$. These morphisms satisfy all the relations satisfied by the corresponding Mackey functor operations, and also the relation that $i_H^K r_H^K$ is multiplication by $|K : H|$ whenever H is a subgroup of K. Further, all relations between the morphisms are deducible from the relations just mentioned. It follows that any R-linear functor $\mathcal{H}_G \to R$-mod can be regarded as a cohomological Mackey functor, and conversely, any cohomological Mackey functor can be regarded as being defined on $\mathcal{H}_G$.

The use of this approach is that any isomorphism between direct sums of permutation modules yields a relationship between the values of a cohomological Mackey functor. Such relationships are exploited in [50], [63] and [8].

The following is an exercise in working from the definitions.

(7.2) PROPOSITION. *Suppose that M is a cohomological Mackey functor and that $H \leq G$ is a subgroup such that $|G : H|$ acts invertibly on all of the R-modules $M(K)$, where $K \leq G$. Then M is projective relative to $\{H\}$.*

In view of this, when $R = \mathbb{Q}$ all cohomological Mackey functors are 1-projective, and when $R = \mathbb{Z}_p$ cohomological Mackey functors are projective relative to the p-subgroups of G. In this case, according to the section on Burnside ring action, they are acted on as the identity by the Burnside ring idempotent f_1.

It is shown in [62] that every cohomological Mackey functor is in fact a homomorphic image of a Mackey functor of the form $M(G) = H^0(G, U)$, where U is a permutation module. In fact, the indecomposable summands of these functors are precisely the indecomposable projective cohomological Mackey functors. Further information about the projective and simple cohomological Mackey functors is given in [62].

8. Globally-defined Mackey functors

We describe now another kind of Mackey functor which has appeared more recently and which appears to be important. These are the globally-defined Mackey functors. In some ways they are more general than the original Mackey functors, and in some ways more restrictive. One main difference is that instead of being defined just on the subgroups of a particular group, they are defined on all finite groups. This is in keeping with many of the natural examples of Mackey functor, such as group cohomology with trivial coefficients, or the character ring, which are in fact defined on all groups. A second main difference is that whereas the original Mackey functors only possess operations corresponding to inclusions of subgroups and conjugations, the globally-defined Mackey functors may possess operations for all group homomorphisms. This possibility necessitates slightly more restrictive axioms to make it work.

Let R be a commutative ring with a 1. By saying that a group K is a *section* of a group G we mean that there is a subgroup H of G and a surjective homomorphism $H \to K$. Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of finite groups satisfying the following two conditions: (1) if G lies in $\mathcal{X}$ and K is a section of G then K lies in $\mathcal{X}$; and (2) if $1 \to A \to B \to C \to 1$ is a short exact sequence of groups with $A \in \mathcal{X}$ and $C \in \mathcal{X}$ then $B \in \mathcal{X}$. We say that a *globally-defined Mackey functor* over R, with respect to $\mathcal{X}$ and $\mathcal{Y}$, is a structure M which specifies an R-module $M(G)$ for each finite group G, together with for each homomorphism $\alpha : G \to K$ with $\operatorname{Ker}\alpha \in \mathcal{Y}$ an R-module homomorphism $\alpha_* : M(G) \to M(K)$ and for each homomorphism $\beta : G \to K$ with $\operatorname{Ker}\beta \in \mathcal{X}$ an R-module homomorphism $\beta^* : M(K) \to M(G)$. These morphisms should satisfy the following relations:

(1) $(\alpha\gamma)_* = \alpha_*\gamma_*$ and $(\beta\delta)^* = \delta^*\beta^*$ always, whenever these are defined;
(2) whenever $\alpha : G \to G$ is an inner automorphism then $\alpha_* = 1 = \alpha^*$;
(3) for every commutative diagram of groups

$$\begin{array}{ccc} G & \xrightarrow{\beta} & H \\ {\scriptstyle\gamma}\uparrow & & \uparrow{\scriptstyle\alpha} \\ \beta^{-1}(K) & \xrightarrow[\delta]{} & K \end{array}$$

in which α and γ are inclusions and β and δ are surjections we have $\alpha^*\beta_* = \delta_*\gamma^*$ whenever $\operatorname{Ker}\beta \in \mathcal{Y}$, and $\beta^*\alpha_* = \gamma_*\delta^*$ whenever $\operatorname{Ker}\beta \in \mathcal{X}$;
(4) for every commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\gamma} & H/\operatorname{Ker}\alpha\operatorname{Ker}\beta \\ {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\delta} \\ H & \xrightarrow[\alpha]{} & K \end{array}$$

in which α, β, γ and δ are all surjections, with $\operatorname{Ker}\beta \in \mathcal{Y}$ and $\operatorname{Ker}\alpha \in \mathcal{X}$, we have $\beta_*\alpha^* = \gamma^*\delta_*$;
(5) (Mackey axiom) for every pair of subgroups $J, K \leq H$ of every group H we have

$$(\iota_K^H)^*(\iota_J^H)_* = \sum_{h\in[K\backslash H/J]} (\iota^K_{K\cap {}^hJ})_* c_{h*} (\iota^J_{K^h\cap J})^*$$

where $\iota_K^H : K \hookrightarrow H$ and $\iota_J^H : J \hookrightarrow H$ etc. are the inclusion maps and $c_h : K^h \cap J \to K \cap {}^hJ$ is the homomorphism $c_h(x) = hxh^{-1}$.

These globally defined Mackey functors form an abelian category denoted $\mathrm{Mack}_R^{\mathcal{X},\mathcal{Y}}$.

At first sight some conditions appear to have been omitted which are necessary to make things work. Thus in both (3) and (4), if $\operatorname{Ker}\beta \in \mathcal{Y}$ it follows that $\operatorname{Ker}\delta \in \mathcal{Y}$ also; in (1), if $\operatorname{Ker}\alpha \in \mathcal{Y}$ and $\operatorname{Ker}\gamma \in \mathcal{Y}$ then $\operatorname{Ker}\alpha\gamma \in \mathcal{Y}$ also; and so on. Axiom (4) implies that if $\alpha : H \to K$ is an isomorphism, then $(\alpha^{-1})_* = \alpha^*$ and $(\alpha^{-1})^* = \alpha_*$. The automorphisms of each group G act on $M(G)$, and because the inner automorphisms act trivially each $M(G)$ has the structure of an $R[\operatorname{Out} G]$-module.

The main reason for having the classes $\mathcal{X}$ and $\mathcal{Y}$ as part of the definition is that a globally-defined Mackey functor need not possess all operations α_* and α^* when α is a surjective group homomorphism, and with each example we discuss the possibilities for $\mathcal{X}$

and $\mathcal{Y}$. It is always possible to take $\mathcal{X}$ and $\mathcal{Y}$ to consist only of the identity group, which is the same as saying that α_* and α^* are only defined when α is injective. Sometimes it is possible to take $\mathcal{X}$ and $\mathcal{Y}$ to be larger classes of groups.

Some of the examples of ordinary Mackey functors we have previously discussed also give examples of globally-defined Mackey functors; and some do not. The following are examples of globally-defined Mackey functors.

- $G_0(kG)$ and $B(G)$: in both these examples we may take $\mathcal{X}$ and $\mathcal{Y}$ to be all finite groups. Whenever $\alpha : G \to H$ is a group homomorphism we may restrict both representations of H and H-sets along α. Also we may form $RH \otimes_{RG} U$ and $H \times_G \Omega$ whenever U is an RG-module and Ω a G-set, and this allows us to form α_*.
- $H^n(G,R)$, $H_n(G,R)$, $\hat{H}^n(G,R)$: the cohomology, homology and Tate cohomology of G in some dimension n with trivial coefficients (arbitrary coefficient modules are not possible since they must be modules for every finite group). For cohomology we may take $\mathcal{X}$ to be all finite groups and $\mathcal{Y}$ to be the identity group. If $\alpha : G \to H$ is a surjective group homomorphism then α^* is inflation. However, provided we allow such inflations it is not possible to define α_* (except on isomorphisms) so as to satisfy the axioms. To see this, consider the fixed point functor $M(G) = H^0(G,R)$ for each finite group G, and also the homomorphisms $1 \xrightarrow{\iota} G \xrightarrow{\beta} 1$. We know that $\iota_* = |G| \cdot \mathrm{id}$ and $\iota^* = \mathrm{id}$. Since $\beta\iota = \mathrm{id}$ we have $\beta_*\iota_* = \mathrm{id}$ and $\iota^*\beta^* = \mathrm{id}$. From this we deduce that $\beta^* = \mathrm{id}$ and $\beta_* = |G|^{-1} \cdot \mathrm{id}$. At this point if $|G|^{-1}$ does not exist in R we see that β_* cannot be defined. Even when $|G|^{-1}$ does exist in R, consider axiom (4) applied to the square
$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ {\scriptstyle\beta}\uparrow & & \uparrow \\ G & \underset{\beta}{\longrightarrow} & 1 \end{array}$$
This allows us to deduce that $\beta_*\beta^* = \mathrm{id}$, that is $|G|^{-1} = 1$, which cannot hold for all finite groups G.
- $K_n(\mathbb{Z}G)$, the algebraic K-theory of $\mathbb{Z}G$. Here we may take $\mathcal{Y} =$ all finite groups but put $\mathcal{X} = 1$ (see [49]), the point being that if $\alpha : G \to H$ is a group homomorphism and P is a projective $\mathbb{Z}G$-module then $\mathbb{Z}H \otimes_{\mathbb{Z}G} P$ is a projective $\mathbb{Z}H$-module, but the restriction of a projective module along a homomorphism α is only projective when α is injective.
- $\mathbb{Q} \otimes D(G)$ when G is a p-group and $D(G)$ is the Dade group of endopermutation kG-modules, where k is a field of characteristic p (see [14]). Here we have to consider a modified version of the theory where we consider functors defined only on p-groups. We may take $\mathcal{X}$ and $\mathcal{Y}$ to be all p-groups.

As is the case with ordinary Mackey functors, there is another definition of globally-defined Mackey functors [10] which is less immediately transparent, but which is usually easier to work with. Given a pair of groups G and H we consider the finite (G,H)-*bisets*.

These are finite sets Ω with a left action of G and a right action of H so that the two actions commute: $g(\omega h) = (g\omega)h$ for all $g \in G$, $h \in H$ and $\omega \in \Omega$. By analogy with the Burnside ring, let $A^{\mathcal{X},\mathcal{Y}}(G,H)$ be the Grothendieck group with respect to disjoint unions of all finite (G,H)-bisets Ω with the property that $\mathrm{Stab}_G(\omega) \in \mathcal{X}$ and $\mathrm{Stab}_H(\omega) \in \mathcal{Y}$ for all $\omega \in \Omega$. This is the free abelian group with the isomorphism classes of transitive such bisets as a basis — we say Ω is *transitive* if given $\omega \in \Omega$, every element of Ω may be written $g\omega h$ for some $g \in G$ and $h \in H$. We now define $A_R^{\mathcal{X},\mathcal{Y}}(G,H) = R \otimes_{\mathbb{Z}} A^{\mathcal{X},\mathcal{Y}}(G,H)$.

Given a third group K there is a product

$$A_R^{\mathcal{X},\mathcal{Y}}(G,H) \times A_R^{\mathcal{X},\mathcal{Y}}(H,K) \to A_R^{\mathcal{X},\mathcal{Y}}(G,K)$$

defined on basis elements as $(\Omega,\Psi) \mapsto \Omega \times_H \Psi$ where the latter amalgamated product is the set of equivalence classes under the relation $(\omega h, \psi) \sim (\omega, h\psi)$ whenever $\omega \in \Omega$, $\psi \in \Psi$ and $h \in H$. This product is associative, and provides in particular a ring structure on $A_R^{\mathcal{X},\mathcal{Y}}(G,G)$. When $\mathcal{X}$ consists of all finite groups and $\mathcal{Y}$ consists of the identity group, this ring is known as the *double Burnside ring* of G, see [3] or [43]. (We have chosen the opposite convention to many authors, who take $\mathcal{X} = 1$ and $\mathcal{Y} =$ all finite groups.)

With all this we associate a category $\mathcal{C}_R^{\mathcal{X},\mathcal{Y}}$ whose objects are all finite groups and where $\mathrm{Hom}_{\mathcal{C}_R^{\mathcal{X},\mathcal{Y}}}(H,G) = A_R^{\mathcal{X},\mathcal{Y}}(G,H)$. The composition of morphisms is the product we have defined, and because we have (apparently perversely!) reversed the order of G and H this composition is correct for applying mappings from the left. Finally, a *globally-defined Mackey functor* (with respect to $\mathcal{X}$ and $\mathcal{Y}$) is an R-linear functor $M : \mathcal{C}_R^{\mathcal{X},\mathcal{Y}} \to R$-mod.

The key to understanding why this definition is equivalent to the first one is to consider for each group homomorphism $\alpha : G \to K$ the bisets ${}_K K_G$ and ${}_G K_K$ where in the first case K acts on K from the left by left multiplication and G acts on K from the right via α and right multiplication, and in the second case the reverse happens. Given a functor M as just defined we define $\alpha_* = M({}_K K_G)$ and $\alpha^* = M({}_G K_K)$. It is the case that these bisets satisfy relations which imply the axioms we have given. Conversely, every transitive biset is a composite of bisets of this special form, and the axioms are sufficient to imply that a Mackey functor defined in the first way gives rise to an R-linear functor $M : \mathcal{C}_R^{\mathcal{X},\mathcal{Y}} \to R$-mod.

9. The computation of globally-defined Mackey functors using simple functors

We describe the first of two applications of globally-defined Mackey functors together with further properties which seem to suggest their importance. The applications depend on a description of the simple globally-defined Mackey functors and we start with this. This material is developed in [66] and [10].

As with ordinary Mackey functors, we can speak of subfunctors of globally-defined Mackey functors, kernels and so forth. We thus have the notion of a simple functor, namely one which has no proper non-zero subfunctors.

(9.1) THEOREM ([10], [66]). *The simple globally-defined Mackey functors are in bijection with pairs (H,U) where H is a finite group and U is a simple $R[\mathrm{Out}\,H]$-module (both taken up to isomorphism). The corresponding simple functor $S_{H,U}$ has the property that $S_{H,U}(H)\cong U$ as $R[\mathrm{Out}\,H]$-modules, and that if G is a group for which $S_{H,U}(G)\neq 0$ then H is a section of G. Provided R is a field or a complete discrete valuation ring each simple functor $S_{H,U}$ has a projective cover $P_{H,U}$, and these form a complete list of the indecomposable projective functors.*

It is a feature of this classification that it is independent of the choice of $\mathcal{X}$ and $\mathcal{Y}$, although the particular structure of the simple functors changes as we vary $\mathcal{X}$ and $\mathcal{Y}$. In the special case when $\mathcal{X}=\mathcal{Y}=1$ an explicit description (stated below) of the values $S_{H,U}(G)$ was given in [66], as well as a less transparent description in the case when $\mathcal{X}$ is all finite groups and $\mathcal{Y}=1$. In this latter case it is shown that the dimension of the $S_{H,U}(G)$ is related to the stable decomposition of the classifying space BG (as will be explained later) and existing computations of these decompositions are really equivalent to computing this dimension. When $\mathcal{X}$ and $\mathcal{Y}$ consist of all finite groups it appears to be rather difficult to describe the simple functors explicitly, in general, but we will return to this question in the last section of this article.

(9.2) THEOREM ([66]). *When $\mathcal{X}=\mathcal{Y}=1$ the simple globally-defined Mackey functors are given explicitly by*

$$S_{H,U}(G)=\bigoplus_{\substack{\alpha:H\cong L\leq G\\ \text{up to } G-\text{conjugacy}}}\mathrm{tr}_L^{N_G(L)}({}^{\alpha}U)$$

where H ranges over finite groups and U ranges over simple $R[\mathrm{Out}(H)]$-modules.

Here the direct sum is taken over G-orbits of isomorphisms α from H to subgroups L of G, and ${}^{\alpha}U$ means U with the action transported to $N_G(L)/L$ via α. The symbol tr means the relative trace, i.e. multiplication by the sum of coset representatives of L in $N_G(L)$.

This straightforward description of the simple functors when $\mathcal{X}=\mathcal{Y}=1$ gives rise to a method of computing the values of globally-defined Mackey functors which has been applied in the case of group cohomology in [66] and [18]. We work with the p-torsion subgroup $M(G) = H^n(G,R)_p$ for a fixed prime p. Although M can be defined with inflation operations α^* when α is a surjective group homomorphism we choose to forget these and regard M as a globally-defined Mackey functor with $\mathcal{X} = \mathcal{Y} = 1$. In the category of such functors we consider a 'composition series' of M, namely a filtration

$$\cdots \subset M_{i-1} \subset M_i \subset M_{i+1} \subset \cdots \subset M$$

such that $\bigcap M_i = 0$, $\bigcup M_i = M$ and M_{i+1}/M_i is always a simple functor. It is shown that such a series exists, and the multiplicity of each simple functor as a factor is determined independently of the choice of the series. Furthermore, the fact that — as an ordinary Mackey functor — cohomology is projective relative to p-subgroups implies that the only simple functors which arise as composition factors in the global situation are $S_{H,U}$ where H is a p-group; and furthermore the multiplicities as composition factors are determined by knowledge of the cohomology of p-groups. Putting all this together, we get a formula for the size of $M(G)$ knowing the composition factor multiplicities and the values $S_{H,U}(G)$, and it is expressed in terms of the cohomology of the p-subgroups of G and conjugacy of p-elements. Given explicit information about the cohomology of the p-subgroups, the formula for the cohomology of G gives completely explicit numerical results. A remarkable feature of this approach is that we obtain a uniform formula which applies at once to all finite groups G with a given Sylow p-subgroup.

10. Stable decompositions of BG

For surveys of the background material to this section see [3] and [43]. We denote by $(BG_+)^\wedge_p$ the p-completion of the suspension spectrum obtained from the classifying space BG after first adjoining a disjoint base point to give a space BG_+. The problem of decomposing $(BG_+)^\wedge_p$ stably as a wedge of indecomposable spectra is a fundamental question which — thanks to Carlsson's proof of the Segal conjecture — comes down to an analysis of the double Burnside ring $A^{\text{all},1}_{\mathbb{Z}_p}(G,G)$ defined in an earlier section. By studying the representations of this ring it was proved by Benson and Feshbach [4] and also by Martino and Priddy [44] that the indecomposable p-complete spectra which can appear as a summand of some $(BG_+)^\wedge_p$ (allowing G to vary over all finite groups) are parametrized by pairs (H,U) where H is a p-group and U is a simple $\mathbb{Z}_p[\mathrm{Out}(H)]$-module. They also gave a method for determining the multiplicity with which each spectrum in the parametrization occurs as a summand of a given BG_+.

We describe in this section how a proof of their theorem may be given entirely within the context of globally-defined Mackey functors (assuming Carlsson's theorem). The complete description can be found in [66]. We work with globally-defined Mackey functors where we take $\mathcal{X}$ to be all finite groups and $\mathcal{Y}$ to be the isomorphism class of the identity group, and we will see that there is an equivalence of categories between the full subcategory of the category of spectra whose objects are the summands of the $(BG_+)^\wedge_p$, and the full subcategory of $\mathrm{Mack}^{\mathcal{X},\mathcal{Y}}_{\mathbb{Z}_p}$ whose objects are the projective covers $P_{H,U}$ of the simple functors $S_{H,U}$ where H is a p-group. The advantage of this approach is that we work with the projective objects in a category — and such a situation is often felt to be well-understood — rather than a more mysterious subcategory of the category of spectra.

For each group K there is a representable functor $F^K = \mathrm{Hom}_{\mathcal{C}^{\mathcal{X},\mathcal{Y}}_{\mathbb{Z}_p}}(K, \ )$ which is a projective object in $\mathrm{Mack}^{\mathcal{X},\mathcal{Y}}_{\mathbb{Z}_p}$ by Yoneda's lemma. This decomposes as a sum of indecomposable projectives, and the particular form of the decomposition will be of use to us in what follows.

(10.1) LEMMA. *The representable functor F^K decomposes as $F^K \cong \bigoplus P_{H,U}^{n_{H,U}}$ where*

$$n_{H,U} = \dim S_{H,U}(K)/\dim \mathrm{End}_{\mathbb{Z}_p[\mathrm{Out}\, H]} U.$$

Thus $P_{H,U}$ is only a summand of F^K if H is a section of K, and $P_{K,U}$ does occur as a summand of F^K with multiplicity $\dim U/\dim \mathrm{End}_{\mathbb{Z}_p[\mathrm{Out}\, H]} U$.

The dimensions are taken over $\mathbb{Z}/p\mathbb{Z}$ here. This is possible since the values of a simple functor over $\mathbb{Z}_p$ are actually $\mathbb{Z}/p\mathbb{Z}$-vector spaces.

Proof. From the properties of a projective cover we have

$$\mathrm{Hom}(P_{H,U}, S_{J,V}) = \begin{cases} \mathrm{End}(S_{H,U}) & \text{if } (H,U) \cong (J,V) \\ 0 & \text{otherwise.} \end{cases}$$

If we write $F^K \cong \bigoplus P_{H,U}^{n_{H,U}}$ for some integers $n_{H,U}$ to be determined, we have

$$\dim \mathrm{Hom}(F^K, S_{H,U}) = n_{H,U} \cdot \dim \mathrm{End}(S_{H,U}).$$

Now

$$\mathrm{Hom}(F^K, S_{H,U}) \cong S_{H,U}(K)$$

by Yoneda's lemma, and also

$$\mathrm{End}(S_{H,U}) \cong \mathrm{End}_{\mathbb{Z}_p[\mathrm{Out}\, H]}(U)$$

from [66] or [10]. Rearranging these equations gives the claimed expression for $n_{H,U}$. We obtain the remaining statements from Theorem 9.1. □

Again by Yoneda's lemma, $\mathrm{Hom}(F^G, F^H) \cong \mathrm{Hom}(H, G) = A_{\mathbb{Z}_p}^{\mathcal{X},\mathcal{Y}}(G, H)$ and composition of morphisms on the left corresponds to the product on the right. It is a consequence of Carlsson's theorem that when P is a p-group we have $[(BP_+)^\wedge_p, (BG_+)^\wedge_p] \cong A_{\mathbb{Z}_p}^{\mathcal{X},\mathcal{Y}}(G, P)$, where the left hand side denotes the homotopy classes of maps of spectra. We have reversed the expected order of G and P on the right hand side so that composition of maps written on the left corresponds to the product of bisets.

We immediately have the first part of the next result.

(10.2) THEOREM. *Let p be a prime.*

(i) The assignment $(BP_+)^\wedge_p \to F^P$ gives an equivalence of categories between the full subcategory of the category of spectra whose objects are the $(BP_+)^\wedge_p$ where P is a p-group, and the full subcategory of $\mathrm{Mack}_{\mathbb{Z}_p}^{\mathcal{X},\mathcal{Y}}$ whose objects are the representable functors F^P.

(ii) The equivalence in (i) extends to an equivalence between the full subcategory of the category of spectra whose objects are stable summands of the classifying spaces $(BG_+)^\wedge_p$ as G ranges over finite groups, and the full subcategory of $\mathrm{Mack}_{\mathbb{Z}_p}^{\mathcal{X},\mathcal{Y}}$ whose objects are the indecomposable projectives $P_{H,U}$ with H a p-group.

Proof. To prove the second part we first observe that the equivalence in part (i) can be extended to summands of the objects, since these correspond to idempotents in the endomorphisms rings of objects, and corresponding objects have isomorphic endomorphism rings. We see from Lemma 10.1 that the indecomposable summands of the F^P with P a p-group are precisely the $P_{H,U}$ with H a p-group. Also it is well known that $(BG_+)^\wedge_p$ is stably a summand of $(BP_+)^\wedge_p$ where P is a Sylow p-subgroup of G, and so the summands of all the $(BG_+)^\wedge_p$ are the same as the summands of all the $(BP_+)^\wedge_p$. This completes the proof. □

We now deduce that the stable summands of the $(BG_+)^\wedge_p$ are parametrized the same way as the $P_{H,U}$ with H a p-group, and the multiplicities of these summands are given by 10.1. The properties given in 10.1 are exactly the properties of the summands of the $(BG_+)^\wedge_p$ given in [4] and [44]. By analyzing the structure of $S_{H,U}(G)$ we are also able to obtain their general formula for these multiplicities.

11. Some naturally-occurring globally-defined Mackey functors

We conclude by pointing out that some very important naturally-occurring Mackey functors are in fact simple in some cases and projective in another. It is remarkable that this highly technical theory encapsulates natural examples in this way. We state results only over $\mathbb{Q}$ but in fact they hold over any field of characteristic 0.

(11.1) THEOREM (Bouc [10], Bouc and Thévenaz [14]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be all finite groups.*

(i) The Burnside ring Mackey functor $\mathbb{Q} \otimes_{\mathbb{Z}} B$ is the indecomposable projective $P_{1,\mathbb{Q}}$.

(ii) The functor $M(G) = \mathbb{Q} \otimes_{\mathbb{Z}} G_0(\mathbb{Q}G)$ which assigns the representation ring of $\mathbb{Q}G$-modules, tensored with $\mathbb{Q}$, is the simple functor $S_{1,\mathbb{Q}}$.

(iii) Let p be a prime. The kernel of the projective cover map $P_{1,\mathbb{Q}} \to S_{1,\mathbb{Q}}$, regarded as a functor only on p-groups, is the functor $\mathbb{Q} \otimes D$, where $D(P)$ is the Dade group of endopermutation modules of the p-group P. This functor is simple: $\mathbb{Q} \otimes D \cong S_{C_p \times C_p,\mathbb{Q}}$.

The first statement in this theorem is straightforward. For each group G we have a representable functor F^G and in case $G = 1$ we may see from the definitions that $F^1 = \mathbb{Q} \otimes_{\mathbb{Z}} B$. We also know the decomposition of this functor as a direct sum of indecomposable projectives, as given in Lemma 10.1, and from this we see it is $P_{1,\mathbb{Q}}$. The second statement is less obvious and appears in [10]. We know from Artin's induction theorem that the natural map $\mathbb{Q} \otimes_{\mathbb{Z}} B \to \mathbb{Q} \otimes_{\mathbb{Z}} G_0(\mathbb{Q}G)$ is surjective. It requires some further argument to show that the target is simple. Statement (iii) is not obvious at all. Dade's group is described in [59] and [14], and we will not discuss it here. By regarding a functor as defined only on p groups, we mean that we are considering the restriction of the functor to the full subcategory of $\mathcal{C}_{\mathbb{Q}}^{\mathcal{X},\mathcal{Y}}$ whose objects are the p-groups, and this restriction is asserted to be simple in the category of functors on this subcategory.

The kernel of the projective cover map $P_{1,\mathbb{Q}} \to S_{1,\mathbb{Q}}$ is not simple as a functor defined on all finite groups, although it is not a straightforward matter to determine its composition factors. We have the following:

(11.2) THEOREM (Bouc [10]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be all finite groups. The composition factors of the Burnside functor $\mathbb{Q} \otimes_{\mathbb{Z}} B$ all have the form $S_{H,\mathbb{Q}}$ for various groups H. Each such simple functor appears with multiplicity at most 1.*

Bouc also characterizes in [10] those groups H for which the simple functor $S_{H,\mathbb{Q}}$ does appear as a composition factor of $\mathbb{Q} \otimes_{\mathbb{Z}} B$ by means of a certain combinatorial condition (he calls such groups 'b-groups'). There is also information in [14] about the composition factors of $k \otimes_{\mathbb{Z}} B$ when k is a field of prime characteristic.

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