The Low-Dimensional Cohomology of Categories

PETER WEBB

1. A RESOLUTION OF $\underline{R}$

The homology and cohomology of a small category $\mathcal{C}$ over a commutative ring R with 1 are defined as $H_*(\mathcal{C}, B) = \mathrm{Tor}_*^{R\mathcal{C}}(B, \underline{R})$ and $H^*(\mathcal{C}, A) = \mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, A)$ where $\underline{R}$ is the constant functor, B is a right $R\mathcal{C}$-module and A is a left $R\mathcal{C}$-module, $R\mathcal{C}$ being the category algebra. For these definitions we refer to [14] and a discussion of the use of representations of categories can be found there as well as in many of the other references we give, such as [1, 2, 3, 4, 6, 9, 10, 13]. The category algebra $R\mathcal{C}$ is the free R-module with the morphisms of $\mathcal{C}$ as a basis, the multiplication being composition of morphisms when defined, and otherwise zero.

Homology and cohomology of $\mathcal{C}$ may be computed by taking a projective resolution of $\underline{R}$, and a resolution analogous to the bar resolution in group cohomology was constructed in [12]. We observe here that a resolution may also be associated to any surjection from a free category (see [11] for the definition) to $\mathcal{C}$ in the same way that in group cohomology one associates the Gruenberg resolution to a presentation of a group. Let $\mathcal{F} \to \mathcal{C}$ be a functor where $\mathcal{F}$ is a free category with the same objects as $\mathcal{C}$, and which is surjective on morphisms. This gives rise to an R-algebra homomorphism $R\mathcal{F} \to R\mathcal{C}$ whose kernel N is a 2-sided ideal in $R\mathcal{F}$. There is a left $R\mathcal{C}$-module homomorphism $R\mathcal{C} \to \underline{R}$ specified on the basis morphisms of $R\mathcal{C}$ by $\alpha \mapsto 1_{\mathrm{cod}\,\alpha}$ where $\mathrm{cod}\,\alpha$ is the codomain of α and we define the *left augmentation ideal* of $R\mathcal{C}$ to be its kernel ${}^LI\mathcal{C}$. Similarly there is a morphism of right modules $R\mathcal{C} \to \underline{R}$ whose kernel is the right augmentation ideal $I\mathcal{C}^R$. By the same arguments as those which establish the Gruenberg resolution [5, p. 34] we have the following.

Theorem 1.1. *With the above notation we have a free resolution of $R\mathcal{C}$-modules*

$$\cdots \to N^2/N^3 \to (N{\cdot}{}^LI\mathcal{F})/(N^2{\cdot}{}^LI\mathcal{F}) \to N/N^2 \to {}^LI\mathcal{F}/(N{\cdot}{}^LI\mathcal{F}) \to R\mathcal{C} \to \underline{R} \to 0.$$

A crucial fact which makes this work is that as left $R\mathcal{F}$-modules, N and ${}^LI\mathcal{F}$ are both free.

We obtain formulas for homology

$$H_{2n}(\mathcal{C}, \underline{R}) \cong (N^n \cap I\mathcal{F}^R \cdot N^{n-1} \cdot {}^LI\mathcal{F})/(I\mathcal{F}^R \cdot N^n + N^n \cdot {}^LI\mathcal{F})$$

and

$$H_{2n+1}(\mathcal{C}, \underline{R}) \cong (I\mathcal{F}^R \cdot N^n \cap N^n \cdot {}^LI\mathcal{F})/(N^{n+1} + I\mathcal{F}^R \cdot N^n \cdot {}^LI\mathcal{F})$$

which fit into a picture exactly exactly like that on [5, p. 48]. In the case of first homology the formula simplifies to give

$$H_1(\mathcal{C}, \underline{R}) \cong (I\mathcal{C}^R \cap {}^LI\mathcal{C})/(I\mathcal{C}^R \cdot {}^LI\mathcal{C})$$

thereby extending the result for groups that the abelianization of the group is isomorphic to the augmentation ideal (over the integers) modulo its square.

2. Relation Modules and Projective Extensions

We will use the notion of an extension of a category which appears in Hoff [8], bearing in mind that there are also other approaches (see [1, 4, 6, 8, 14]).

Starting with any surjective functor $\mathcal{F} \to \mathcal{C}$ where $\mathcal{F}$ is a free category with the same objects as $\mathcal{C}$ we construct a relation module with similar properties to relation modules coming from presentations of groups. With categories we do not have a satisfactory notion of the kernel of the functor, and it is hard to see how to proceed by taking the abelianization of the kernel as in group theory. Instead we define the *relation module* to be the $R\mathcal{C}$-module $N/(N \cdot {}^L I\mathcal{F})$, and it appears in a short exact sequence of $R\mathcal{C}$-modules $0 \to N/(N \cdot {}^L I\mathcal{F}) \to R\mathcal{C}^d \to {}^L I\mathcal{C} \to 0$, which is part of the resolution we have constructed. It has the property that it occurs as the left hand term in an extension of the category $\mathcal{C}$ which is projective in a certain category of such extensions, in the manner of [5, p. 197]. To construct this extension observe that $H^2(\mathcal{C}, A) \cong \operatorname{Ext}^1_{R\mathcal{C}}({}^L I\mathcal{C}, A)$ for all modules A, so that the short exact sequence of $R\mathcal{C}$-modules just mentioned corresponds to a category extension of $\mathcal{C}$ by $N/(N \cdot {}^L I\mathcal{F})$, and this has the projective property among extensions of $\mathcal{C}$ by an $R\mathcal{C}$-module because the short exact sequence of $R\mathcal{C}$-modules is projective among extensions of ${}^L I\mathcal{C}$.

3. Five term exact sequences

By following the proof of the analogous result for group cohomology on [7, p. 202] we obtain the following.

Theorem 3.1. *Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension of categories, let B be a right $R\mathcal{C}$-module and let A a left $R\mathcal{C}$-module. There are exact sequences*

$$H_2(\mathcal{E}, B) \to H_2(\mathcal{C}, B) \to B \otimes_{R\mathcal{C}} H_1(\mathcal{K}) \to H_1(\mathcal{E}, B) \to H_1(\mathcal{C}, B) \to 0$$

and

$$H^2(\mathcal{E}, A) \leftarrow H^2(\mathcal{C}, A) \leftarrow \operatorname{Hom}_{R\mathcal{C}}(H_1(\mathcal{K}), A) \leftarrow H^1(\mathcal{E}, A) \leftarrow H^1(\mathcal{C}, A) \leftarrow 0.$$

In this result $H_1(\mathcal{K}) = H_1(\mathcal{K}, \underline{\mathbb{Z}}) \cong H_1(|\mathcal{K}|)$ is the product of the abelianizations of the groups $K(x)$ which are the automorphism groups of the objects x of $\mathcal{K}$, and because we have an extension of categories it becomes a left $R\mathcal{C}$-module.

We may define the *Schur multiplier* of $\mathcal{C}$ to be $H_2(\mathcal{C}, \underline{\mathbb{Z}})$, and it has a property which generalizes a corresponding result for groups.

Theorem 3.2. *Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension of categories and suppose that the induced homomorphism $H_1(\mathcal{E}) \to H_1(\mathcal{C})$ is an isomorphism. Then $\varinjlim H_1(\mathcal{K}) = \underline{\mathbb{Z}} \otimes_{\mathbb{Z}\mathcal{C}} H_1(\mathcal{K})$ is a homomorphic image of $H_2(\mathcal{C}, \mathbb{Z})$.*

For groups the corresponding theorem is sometimes stated for central extensions with the property that the normal subgroup is contained in the derived subgroup of the extension group, such extensions being called *stem extensions*. With categories we replace this condition by the stated hypothesis on H_1.

4. Lyndon-Hochschild-Serre Spectral Sequences

In group cohomology the five term sequences may be viewed as coming from the Lyndon-Hochschild-Serre spectral sequences, and we ask if the same is true for categories in general. This is not entirely clear, and we report on a result of Fei Xu [15].

Theorem 4.1. *Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be such that the corresponding functors between the opposite categories are an extension of the opposite category $\mathcal{C}^{\mathrm{op}}$. Let B be a right $R\mathcal{E}$-module and A a left $R\mathcal{E}$-module. There are spectral sequences whose second pages are*

$$E^2_{p,q} = H_p(\mathcal{C}, H_q(\mathcal{K}, B)) \Rightarrow H_{p+q}(\mathcal{E}, B)$$

and

$$E_2^{p,q} = H^p(\mathcal{C}, H^q(\mathcal{K}, A)) \Rightarrow H^{p+q}(\mathcal{E}, A).$$

The fact that we have an extension of $\mathcal{C}^{\mathrm{op}}$ is here used to make $H_q(\mathcal{K}, B)$ into a right $R\mathcal{C}$-module, and $H^q(\mathcal{K}, A)$ into a left $R\mathcal{C}$-module, and this is not the same dependence as the one which appeared in the five term exact sequences presented in the previous section.

References

[1] H.-J. Baues and G. Wirsching, *Cohomology of small categories*, J. Pure Appl. Algebra **38** (1985), 187–211.

[2] C. Broto, R. Levi and B. Oliver, *The theory of p-local groups: a survey*, preprint March 2005.

[3] W.G. Dwyer and H.W. Henn, *Homotopy theoretic methods in group cohomology*, Advanced Courses in Mathematics–CRM Barcelona, Birkhauser Verlag, Basel, 2001.

[4] A.I. Generalov, *Relative homological algebra, cohomology of categories, posets and coalgebras*, pp. 611–638 in M. Hazewinkel (ed.), Handbook of Algebra, Vol. I, Elsevier 1996.

[5] K.W. Gruenberg, *Cohomological topics in group theory*, Lecture Notes in Mathematics **143**, Springer-Verlag (1970).

[6] A. Haefliger, *Extension of complexes of groups*, Ann. Inst. Fourier (Grenoble) **42** (1992), 275–311.

[7] P.J. Hilton and U. Stammbach, *A course in homological algebra*, Graduate Texts in Math. **4**, Springer-Verlag 1970.

[8] G. Hoff, *Cohomologies et extensions de categories*, Math. Scand. **74** (1994), 191–207.

[9] M. Linckelmann, *Fusion category algebras*, J. Algebra **277** (2004), 222–235.

[10] M. Linckelmann, *Alperin's weight conjecture in terms of equivariant Bredon cohomology*, Math. Z. **250** (2005), 495–513.

[11] S. MacLane, *Categories for the working mathematician*, Grad. Texts in Math. **5**, Springer-Verlag, New York, 1971.

[12] J.-E. Roos, *Sur les foncteurs dérivées de* $\varprojlim$, Comptes Rendues Acad. Sci. Paris **252** (1961), 3702–3704.

[13] P.J. Webb, *Standard stratifications of EI categories and Alperin's weight conjecture*, preprint 2005, available from http://www.math.umn.edu/∼webb.

[14] P.J. Webb, *An introduction to the representations and cohomology of categories*, to appear in proceedings of the Bernoulli Centre program on group representations, Lausanne 2005, preprint available from http://www.math.umn.edu/∼webb.

[15] F. Xu, *Homological properties of category algebras*, Ph.D. thesis, University of Minnesota 2006.