Simple Mackey Functors

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In connection with recent developments in group representation theory which make use of the theory of Mackey functors [T-W], the natural question of the classification of *simple* Mackey functors arose. The purpose of the present paper is to give a complete answer to this question. For applications, the reader can refer to our paper [T-W] where our main results are actually used in an essential way.

After recalling the definitions and elementary facts about Mackey functors, we prove in Section 2 that every simple Mackey functor S for a finite group G has (up to conjugation) a unique minimal subgroup H with $S(H) \neq 0$, and that the $(N_G(H)/H)$-module $V = S(H)$ is simple. Similar methods easily give a criterion for the simplicity of a Mackey functor (Section 3). We attach to S the pair (H, V) just described and it turns out that this provides a parameterisation of the simple Mackey functors. In order to prove this, we describe explicitly a simple Mackey functor corresponding to an arbitrary pair (H, V). This requires a number of constructions which are introduced in Sections 4, 5 and 6, namely restriction, induction, inflation of Mackey functors and fixed point functors. All of these constructions are very natural and have useful adjointness properties. Sections 7 and 8 are devoted to the classification of simple Mackey functors. In the last section, we prove (in two different ways) that over a field whose characteristic is zero or prime to $|G|$, any Mackey functor is semi-simple, that is, a direct sum of simple Mackey functors.

Mackey functors for G with a multiplicative structure are called Green functors or simply G-functors. We note here that the classification of simple G-functors appears in [T 2]. Our main result is similar, but the methods are quite different.

Notations. Throughout the paper, G denotes a finite group. We write $H \leq G$ (resp. $H < G$) to indicate that H is a subgroup of G (resp. a proper subgroup of G). The notation $K =_G H$ means that K is G-conjugate to H. Similarly we write $K \leq_G H$ (resp. $K <_G H$) when K is G-conjugate to a subgroup of H (resp. a proper subgroup of H). If $H, K \leq G$, we denote by $[G/H]$ (resp. $[K\backslash G/H]$) an arbitrary set of representatives of the cosets gH (resp. the double cosets KgH). Finally we set $\overline{N}_G(H) = N_G(H)/H$.

1. Definitions

As a base ring, we fix a commutative ring R (which will later be a field). Let G be a finite group and $\mathcal{X}$ a family of subgroups of G closed under taking subgroups and conjugation (in short, subconjugacy closed). Our main interest is in the family of all subgroups of G, but it is convenient to give the following more general definition. A *Mackey functor* M on $\mathcal{X}$ (over R) is a family of R-modules $M(H)$ where H runs through the set $\mathcal{X}$, together with *restriction* maps $r^H_K : M(H) \to M(K)$, *transfer* maps $t^H_K : M(K) \to M(H)$ (where in both cases $K \leq H$) and *conjugation* maps $c_g : M(H) \to M({}^gH)$ where $g \in G$ and ${}^gH = gHg^{-1}$, such that the following axioms are satisfied. For any $g, h \in G$ and $H, K, L \in \mathcal{X}$,

(i) if $K \leq H$, $\; r^H_K$, t^H_K and c_g are R-linear maps.

(ii) if $L \leq K \leq H$, $\; r^K_L\, r^H_K = r^H_L$, $\; t^H_K\, t^K_L = t^H_L$; moreover $r^H_H = t^H_H = id_{M(H)}$.

(iii) $c_{gh} = c_g c_h$.

(iv) if $K \leq H$, $\; c_g\, r^H_K = r^{{}^gH}_{{}^gK}\, c_g \;$ and $\; c_g\, t^H_K = t^{{}^gH}_{{}^gK}\, c_g$.

(v) if $h \in H$, $\; c_h : M(H) \to M(H)\;$ is the identity.

(vi) (Mackey axiom) if $L, K \leq H$,

$$r^H_L\, t^H_K = \sum_{g \in [L\backslash H/K]} t^L_{L\cap {}^gK}\, r^{{}^gK}_{L\cap {}^gK}\, c_g$$

where $[L\backslash H/K]$ denotes an arbitrary set of representatives of the double cosets LgK.

When $\mathcal{X}$ is the family of all subgroups of G, we shall simply say that M is a Mackey functor for G.

Note that the axioms (iii) and (v) imply that $\overline{N}_G(H) = N_G(H)/H$ acts on $M(H)$ (via R-linear automorphisms), so that $M(H)$ is an $R\overline{N}_G(H)$-module. In particular $M(1)$ is an RG-module. This implies that the category of RG-modules is embedded in the category of Mackey functors (see Section 6).

There is an equivalent definition in terms of finite G-sets (see [Dr] or [t D, Section 6.1] for details). The value of M on the transitive G-set G/H is simply $M(G/H) = M(H)$ and then M is also defined on disjoint unions of transitive G-sets thanks to the formula $M(S \cup T) = M(S) \oplus M(T)$. In order to deal with morphisms, we note that we have the following G-homomorphisms between transitive G-sets: if $K \leq H$ are subgroups of G then

$$\pi^H_K : G/K \to G/H \quad ; \quad gK \mapsto gH$$

denotes the natural quotient map, and for $x \in G$

$$c_x : G/H \to G/{}^xH \quad ; \quad gH \mapsto (gx^{-1})\,{}^xH$$

denotes conjugation by x. Then any G-homomorphism between two transitive G-sets is a composite of a natural quotient map and a conjugation. Now it turns out that one can define a Mackey functor as a bifunctor (M_*, M^*) on the category of G-sets with very few defining properties. By definition M_* is covariant and M^* is contravariant. Moreover M_* and M^* coincide on objects and we write $M(S) = M_*(S) = M^*(S)$. The link with the previous definition is provided by the following dictionary:

$$M_*(\pi_K^H) = t_K^H : M(K) \to M(H)$$
$$M^*(\pi_K^H) = r_K^H : M(H) \to M(K)$$
$$M_*(c_x) = c_x : M(H) \to M({}^xH)$$
$$M^*(c_x) = c_x^{-1} : M({}^xH) \to M(H).$$

This point of view is extremely fruitful but we shall only occasionnally use it, for the elementary approach suffices for our purposes.

Recall that a *homomorphism* of Mackey functors $f : M \to N$ is a family of R-module homomorphisms $f(H) : M(H) \to N(H)$ (where H runs in $\mathcal{X}$) which commute with restriction, transfer and conjugation (in the obvious sense). Since f commutes with conjugation, $f(H)$ is in particular an $R\overline{N}_G(H)$-linear homomorphism. The set of homomorphisms from M to N is written $\mathrm{Hom}_{\mathrm{Mack}(\mathcal{X})}(M, N)$ (respectively $\mathrm{Hom}_{\mathrm{Mack}(G)}(M, N)$ when $\mathcal{X}$ is the family of all subgroups of G). The category of Mackey functors for G is written $\mathrm{Mack}(G)$.

By a *subfunctor* N of a Mackey functor M on $\mathcal{X}$, one means a family of R-submodules $N(H) \subseteq M(H)$ (where $H \in \mathcal{X}$) which is stable under restriction, transfer and conjugation (in the obvious sense). If N is a subfunctor of M, then the *quotient functor* M/N is defined by $(M/N)(H) = M(H)/N(H)$, with the induced restriction, transfer and conjugation maps. It is not difficult to check that the category of Mackey functors is an abelian category. Finally a Mackey functor M is called *simple* (or *irreducible*) if the only subfunctors of M are M itself and 0 (i.e. the Mackey functor which is 0 on each subgroup). The set of isomorphism classes of simple Mackey functors for G is written $\mathrm{Irr}_R(\mathrm{Mack}(G))$.

Let I be a maximal ideal of R with residue field k and let M be a Mackey functor for G. Since all restriction, transfer and conjugation maps are R-linear, it is clear that the family of k-vector spaces

$$(k \otimes_R M)(H) = k \otimes_R M(H) = M(H) \,/\, I{\cdot}M(H)$$

is endowed with a structure of Mackey functor for G over k. If M is simple, then $k \otimes_R M$ is either equal to M or to 0, but it cannot be zero for all maximal ideals I of R (by standard commutative algebra). Therefore we have

$$\mathrm{Irr}_R(\mathrm{Mack}(G)) = \bigcup_{I \text{ maximal}} \mathrm{Irr}_{R/I}(\mathrm{Mack}(G))$$

and it does no harm to assume that $R = k$ is a field. We shall make this assumption for the rest of the paper. However we note that we never actually use this assumption (except in Section 9); it is just that it seems to us misleading not to emphasize that simple Mackey functors are naturally defined over a field.

2. The parameterisation of simple Mackey functors

Let M be a Mackey functor for G over a field k and for each subgroup H of G let $E(H)$ be a subset of $M(H)$. We put

$$<E>= \bigcap \{N \mid N \text{ is a subfunctor of } M \text{ with } E(H) \subseteq N(H) \text{ for all subgroups } H\}.$$

We call $<E>$ the *subfunctor generated by* E. More generally, let $\mathcal{F}$ be any family of subgroups of G and for each $H \in \mathcal{F}$, let $E(H)$ be a subset of $M(H)$; by the subfunctor generated by E, we mean the subfunctor generated by $\tilde{E}$ where $\tilde{E}$ is the extension of the family E to all subgroups of G defined by $\tilde{E}(H) = 0$ if $H \notin \mathcal{F}$. If M is a Mackey functor for G and if $\mathcal{X}$ is a subconjugacy closed family of subgroups of G, we denote by $M\downarrow_{\mathcal{X}}$ the restriction of M to $\mathcal{X}$, viewed as a Mackey functor defined on $\mathcal{X}$.

(2.1) PROPOSITION. *Let M be a Mackey functor for G. Let $\mathcal{X}$ be a subconjugacy closed family of subgroups of G and N a subfunctor of $M\downarrow_{\mathcal{X}}$ (defined on $\mathcal{X}$). Then for all subgroups K,*

$$<N>(K) = \sum_{\substack{X \in \mathcal{X} \\ X \leq K}} t_X^K N(X).$$

If $K \in \mathcal{X}$ then $<N>(K) = N(K)$, or in other words $<N>\downarrow_{\mathcal{X}} = N$.

Proof. Let

$$L(K) = \sum_{\substack{X \in \mathcal{X} \\ X \leq K}} t_X^K N(X).$$

We check that L is indeed a subfunctor of M. We have to prove that L is closed under restriction, transfer and conjugation. Closure under transfer and conjugation are evident. Closure under restriction follows from

$$\begin{aligned} r_J^K L(K) &= \sum_{\substack{X \in \mathcal{X} \\ X \leq K}} r_J^K t_X^K N(X) \\ &= \sum_{\substack{X \in \mathcal{X} \\ X \leq K}} \sum_{g \in [J \backslash K / X]} t_{{}^gX \cap J}^J \, c_g \, r_{X \cap {}^{g^{-1}}J}^X N(X) \\ &\subseteq \sum_{\substack{Y \in \mathcal{X} \\ Y \leq J}} t_Y^J N(Y) \\ &= L(J) \end{aligned}$$

It is now apparent that any subfunctor of M "containing" N must contain L, so we obtain that $L = <N>$.

Finally if $K \in \mathcal{X}$ we have

$$L(K) = \sum_{\substack{X \in \mathcal{X} \\ X \leq K}} t_X^K N(X) = N(K)$$

since $N(K) = t_K^K N(K)$ is one of the terms, and it contains all the others. □

(2.2) COROLLARY. *Let S be a simple Mackey functor for G and $\mathcal{X}$ a subconjugacy closed set of subgroups of G. Then $S\downarrow_{\mathcal{X}}$ is either a simple Mackey functor on $\mathcal{X}$ or is zero.*

Proof. Suppose N is a non-zero subfunctor of $S\downarrow_{\mathcal{X}}$. This generates a non-zero subfunctor $<N>$ of S, and so $<N> = S$ by simplicity of S. Therefore by the proposition above, $N = <N> \downarrow_{\mathcal{X}} = S\downarrow_{\mathcal{X}}$, proving that $S\downarrow_{\mathcal{X}}$ is simple. □

(2.3) PROPOSITION. *Let S be a simple Mackey functor for G and let H be a minimal subgroup such that $S(H) \neq 0$.*

(i) S is generated by $S(H)$.

(ii) The conjugacy class of H is the unique conjugacy class of minimal subgroups K such that $S(K) \neq 0$.

(iii) $S(H)$ is a simple $k\overline{N}_G(H)$-module.

Proof. (i) is clear because, by simplicity, S is generated by any non-zero family of subsets (a single subset in our case). Let $\mathcal{X}$ consist of all conjugates of subgroups of H. Let W be a non-zero $k\overline{N}_G(H)$-submodule of $S(H)$ and for $K \in \mathcal{X}$, let

$$E(K) = \begin{cases} {}^gW & \text{if } K = {}^gH, \\ 0 & \text{if } K <_G H. \end{cases}$$

It is clear that E is a non-zero subfunctor of $S\downarrow_{\mathcal{X}}$, because since S vanishes on subgroups $K <_G H$, no restriction and transfer are involved, while closure under conjugation is clear since W is an $k\overline{N}_G(H)$-submodule. Then $<E>$ is a non-zero subfunctor of S, so equals S because of simplicity. But from the description in the last proposition, $<E>(K) \neq 0$ only if K contains a conjugate of H, which proves (ii). Moreover the second statement in Proposition 2.1 gives $S(H) = <E>(H) = E(H) = W$ which establishes the simplicity of $S(H)$. □

A minimal subgroup H such that $S(H) \neq 0$ will be called an *minimal subgroup* of S. This definition is given for an arbitrary Mackey functor S, but when S is simple, then the minimal subgroups of S are all conjugate.

Let Ω be the set of pairs (H, V) where H is a subgroup of G and V a simple $k\overline{N}_G(H)$-module (up to isomorphism). The group G acts by conjugation on Ω and we are interested in the set of orbits Ω/G. These are represented by pairs (H, V) where H is taken up to G-conjugation and V is taken up to $k\overline{N}_G(H)$-isomorphism. Proposition 2.3 allows to define a map

$$\Phi : \mathrm{Irr}_k(\mathrm{Mack}(G)) \longrightarrow \Omega/G$$

sending S to the G-orbit of (H, V) where H is an minimal subgroup of S and $V = S(H)$. The first goal of this paper is to prove the following result.

(2.4) THEOREM. *The map Φ is a bijection.*

In fact our second goal is to construct explicitly an inverse of Φ, that is, to construct for each pair $(H, V) \in \Omega$ a simple Mackey functor $S_{H,V}$ with minimal subgroup H and with $S_{H,V}(H) = V$. This will be done in Theorem 8.3 and will complete the classification of simple Mackey functors.

3. A characterization of simple Mackey functors

Before embarking on the preparatory work for the classification, we give a characterization of simple Mackey functors which is based only on the elementary methods of the previous section.

By Proposition 2.1, if $\mathcal{X}$ is a family of subgroups closed under subconjugation, then the subfunctor $< M \downarrow_{\mathcal{X}} >$ generated by the values of a Mackey functor M on $\mathcal{X}$ is equal to the sum of the images of the transfer maps from subgroups in $\mathcal{X}$. Let us denote by $\mathrm{Im}\, t_{\mathcal{X}}$ this subfunctor. There is a dual notion using kernels of restrictions: we denote by $\mathrm{Ker}\, r_{\mathcal{X}}$ the subfunctor of M defined by

$$(\mathrm{Ker}\, r_{\mathcal{X}})(K) = \bigcap_{\substack{X \in \mathcal{X} \\ X \leq K}} \mathrm{Ker}\, r^K_X \ .$$

One checks that $\mathrm{Ker}\, r_{\mathcal{X}}$ is indeed a subfunctor in a fashion similar to the proof of Proposition 2.1.

Now let H be a minimal subgroup of M and take $\mathcal{X}$ to be the subconjugacy closure of H. Then only the conjugates of H actually come into play and we have

$$(\mathrm{Im}\, t_{\mathcal{X}})(K) = \sum_{\substack{g \in G \\ {}^gH \leq K}} \mathrm{Im}\, t^K_{{}^gH} \ ,$$

$$(\mathrm{Ker}\, r_{\mathcal{X}})(K) = \bigcap_{\substack{g \in G \\ {}^gH \leq K}} \mathrm{Ker}\, r^K_{{}^gH} \ .$$

We use this notation for the following characterization of simple Mackey functors.

(3.1) THEOREM. *Let S be a Mackey functor for G, let H be a minimal subgroup of S and let $\mathcal{X}$ be the subconjugacy closure of H. Then S is a simple Mackey functor if and only if the following three conditions are satisfied.*

(i) $\operatorname{Ker} r_{\mathcal{X}} = 0$,

(ii) $\operatorname{Im} t_{\mathcal{X}} = S$,

(iii) $S(H)$ *is a simple* $k\overline{N}_G(H)$*-module.*

Proof. Assume S is simple. Since $(\operatorname{Ker} r_{\mathcal{X}})(H) = 0$, $\operatorname{Ker} r_{\mathcal{X}}$ is a proper subfunctor, hence is zero. The other two conditions have been already proved in (2.3).

Assume now that the three conditions hold and let T be a non-zero subfunctor of S. Then there exists a subgroup K such that $T(K) \neq 0$. Now by (i)

$$(\operatorname{Ker} r_{\mathcal{X}})(K) = \bigcap_{\substack{g \in G \\ {}^gH \leq K}} \operatorname{Ker} r^K_{{}^gH} = 0$$

and therefore there exists $g \in G$ such that ${}^gH \leq K$ and $T(K) \not\subseteq \operatorname{Ker} r^K_{{}^gH}$. It follows that $r^K_{{}^gH}(T(K)) \neq 0$ and so $T({}^gH) \neq 0$. By conjugation, we obtain $T(H) \neq 0$. But since T is a subfunctor of S, $T(H)$ is a $k\overline{N}_G(H)$-submodule of $S(H)$ and so $T(H) = S(H)$ by (iii). By conjugation, we now have $T({}^xH) = S({}^xH)$ for all $x \in G$. By (ii) we deduce

$$S(K) = (\operatorname{Im} t_{\mathcal{X}})(K) = \sum_{\substack{x \in G \\ {}^xH \leq K}} t^K_{{}^xH}\, T({}^xH) \subseteq T(K)$$

and this proves that $T = S$. □

Note that since S vanishes on proper subgroups of H, both conditions (i) and (ii) imply that $S(K) = 0$ if K does not contain a conjugate of H, that is, H is the unique minimal subgroup of S up to conjugation.

In the same vein as the proposition above, we prove a related result which will be used in Section 9.

(3.2) PROPOSITION. *Let $\mathcal{X}$ be a family of subgroups of G closed under subconjugation. Let M be a Mackey functor for G satisfying $M = \operatorname{Im} t_{\mathcal{X}}$ and $\operatorname{Ker} r_{\mathcal{X}} = 0$. If the Mackey functor $M\!\downarrow_{\mathcal{X}}$ on $\mathcal{X}$ decomposes as a direct sum of Mackey functors $M\!\downarrow_{\mathcal{X}} = \bigoplus_i N_i$, then M decomposes as $M = \bigoplus_i M_i$, where $M_i = \langle N_i \rangle$. Moreover $N_i = (M_i)\!\downarrow_{\mathcal{X}}$.*

Proof. Let $M_i = \langle N_i \rangle$. By Proposition 2.1, $(M_i)\!\downarrow_{\mathcal{X}} = N_i$. Now since M is generated by $M\!\downarrow_{\mathcal{X}}$, it is clear that $M = \sum_i M_i$ and we have to show that this sum is direct. Let $m_i \in M_i(K)$ with $\sum_i m_i = 0$. Then for every $X \in \mathcal{X}$ with $X \leq K$, we have $\sum_i r^K_X(m_i) = 0$. Since $r^K_X(m_i) \in (M_i)\!\downarrow_{\mathcal{X}}(X) = N_i(X)$, we obtain $r^K_X(m_i) = 0$ and therefore $m_i \in (\operatorname{Ker} r_{\mathcal{X}})(K) = 0$. □

4. Restriction and induction.

In this section, we recall the definition of restriction and induction for Mackey functors. We first use the approach using G-sets, which is particularly convenient here, but we then translate the definitions in terms of the elementary approach.

Let H be a subgroup of G. There is an obvious restriction functor from the category of (left) G-sets to the category of H-sets, mapping a G-set X to the H-set $X\downarrow_H^G$, defined to be X with the restriction to H of the action of G. There is also an induction functor mapping an H-set Y to the G-set $Y\uparrow_H^G = G\times_H Y$, defined to be the quotient of $G\times Y$ by the equivalence relation $\sim$, where $(gh,y)\sim(g,hy)$ for every $h\in H$. Induction behaves very well on transitive H-sets, for $(H/K)\uparrow_H^G\cong G/K$.

(4.1) LEMMA. *The induction functor* $\uparrow_H^G$: *H-sets* $\to$ *G-sets is left adjoint to the restriction functor* $\downarrow_H^G$: *G-sets* $\to$ *H-sets .*

Proof. This is standard. The counit and unit of the adjunction are

$$\epsilon : (G\times_H X\downarrow_H^G)\to X \quad \text{and} \quad \eta : Y\to (G\times_H Y)\downarrow_H^G \ .$$

Here ϵ maps the class of (g,x) to gx and η maps y to the class of $(1,y)$. $\square$

It should be noted that $\uparrow_H^G$ is not the right adjoint of $\downarrow_H^G$ in the category of G-sets.

Now letting $\mathrm{Mack}(G)$ denote the category of Mackey functors for G we have restriction and induction functors

$$\downarrow_H^G : \mathrm{Mack}(G)\to\mathrm{Mack}(H)$$

$$\uparrow_H^G : \mathrm{Mack}(H)\to\mathrm{Mack}(G)$$

defined (on G-sets) by

$$M\downarrow_H^G(Y) = M(Y\uparrow_H^G).$$

$$M\uparrow_H^G(X) = M(X\downarrow_H^G).$$

These functors satisfy relationships inherited from the corresponding operations for G-sets.

(4.2) PROPOSITION. *The induction functor* $\uparrow_H^G\colon \mathrm{Mack}(H) \to \mathrm{Mack}(G)$ *is both the left and the right adjoint of the restriction functor* $\downarrow_H^G\colon \mathrm{Mack}(G) \to \mathrm{Mack}(H)$.

Proof. First we show that $\uparrow_H^G$ is left adjoint to $\downarrow_H^G$, that is we show that there is a natural bijection

$$\mathrm{Hom}_{\mathrm{Mack}(G)}(M \uparrow_H^G, N) \cong \mathrm{Hom}_{\mathrm{Mack}(H)}(M, N \downarrow_H^G).$$

We do this by defining the counit and unit of the proposed adjunction

$$p(N) : N \downarrow_H^G \uparrow_H^G \to N \qquad\qquad q(M) : M \to M \uparrow_H^G \downarrow_H^G$$

so as to be natural in M and N. We specify these natural transformations by defining their evaluation on each G-set X and H-set Y as follows:

$$p(N) = N_*(\epsilon) \;:\; N(X \downarrow_H^G \uparrow_H^G) = N \downarrow_H^G \uparrow_H^G (X) \longrightarrow N(X)$$

$$q(M) = M_*(\eta) \;:\; M(Y) \longrightarrow M(Y \uparrow_H^G \downarrow_H^G) = M \uparrow_H^G \downarrow_H^G (Y)$$

where ϵ and η are the maps defined in Lemma 4.1, and where N_* denotes the covariant functor which is part of the definition of the Mackey functor N (and similarly for M_*).

Naturality of p and q with respect to M and N follows from the naturality of ϵ and η. To show that they give an adjunction we have to verify that the composites

$$N \downarrow_H^G \xrightarrow{q(N\downarrow_H^G)} N \downarrow_H^G \uparrow_H^G \downarrow_H^G \xrightarrow{p(N)\downarrow_H^G} N \downarrow_H^G$$

$$M \uparrow_H^G \xrightarrow{q(M)\uparrow_H^G} M \uparrow_H^G \downarrow_H^G \uparrow_H^G \xrightarrow{p(M\uparrow_H^G)} M \uparrow_H^G$$

are the identities $1_{N\downarrow_H^G}$ and $1_{M\uparrow_H^G}$ respectively. This is so because these composites are N_* and M_* applied to

$$Y \uparrow_H^G \xrightarrow{\eta(Y)\uparrow_H^G} Y \uparrow_H^G \downarrow_H^G \uparrow_H^G \xrightarrow{\epsilon(Y\uparrow_H^G)} Y \uparrow_H^G$$

$$X \downarrow_H^G \xrightarrow{\eta(X\downarrow_H^G)} X \downarrow_H^G \uparrow_H^G \downarrow_H^G \xrightarrow{\epsilon(X)\downarrow_H^G} X \downarrow_H^G$$

respectively. These last two composites are the identity in both cases because ϵ and η are the counit and unit of the adjunction between induction and restriction of G-sets.

The proof that $\uparrow_H^G$ is right adjoint to $\downarrow_H^G$ is very similar. We define the counit and unit of the adjunction to be

$$u(M) : M \uparrow_H^G \downarrow_H^G \to M \qquad\qquad v(N) : N \to N \downarrow_H^G \uparrow_H^G$$

$$u(M) = M^*(\eta) \;:\; M \uparrow_H^G \downarrow_H^G (Y) = M(Y \uparrow_H^G \downarrow_H^G) \longrightarrow M(Y)$$

$$v(N) = N^*(\epsilon) \;:\; N(X) \longrightarrow N \downarrow_H^G \uparrow_H^G (X) = N(X \downarrow_H^G \uparrow_H^G).$$

The verification that this gives an adjunction proceeds exactly as before, except now we work with M^* and N^* instead of M_* and N_*. □

Now we translate the above constructions in terms of the elementary approach to Mackey functors. First since $(H/K)\uparrow_H^G = G/K$, the definition of restriction gives

$$N\downarrow_H^G (K) = N\downarrow_H^G (H/K) = N((H/K)\uparrow_H^G) = N(G/K) = N(K)$$

and therefore the restriction of Mackey functors consists simply in restricting our attention to subgroups of H. This is the obvious restriction procedure one would expect. Induction however is more involved.

(4.3) PROPOSITION. *Let M be a Mackey functor for the subgroup H of G. Then*

$$M\uparrow_H^G (K) \cong \bigoplus_{g\in[K\backslash G/H]} M(H\cap K^g)\,.$$

Writing x_g for the component in $M(H\cap K^g)$ of an element $x \in M\uparrow_H^G (K)$ and letting $L \leq K$, $y \in M\uparrow_H^G (L)$, $x \in M\uparrow_H^G (K)$ and $s \in G$, we have

$$\begin{aligned} r_L^K(x)_g &= r_{H\cap L^g}^{H\cap K^g}(x_g) \\ t_L^K(y)_g &= \sum_{u\in[L\backslash K/K\cap {}^gH]} t_{H\cap L^{ug}}^{H\cap K^{ug}}(y_{ug}) \;. \\ c_s(x)_g &= x_{s^{-1}g} \end{aligned}$$

The proof of the proposition is left as an exercise for the reader. Note that one can also *define* an induced functor using the formulae in (4.3), and then check the axioms. This is Sasaki's approach in [Sa, Definition 2.9], attributed to Yoshida. Proposition 4.3 is Lemma 2.2 of that paper. See also [T 2, Section 11].

We end this section with a trivial but useful result.

(4.4) PROPOSITION. *The induction functor is an exact functor. In particular if $L \to M$ is a monomorphism (resp. epimorphism) of Mackey functors for H, then the induced morphism $L\uparrow_H^G \to M\uparrow_H^G$ is a monomorphism (resp. epimorphism) of Mackey functors for G.*

Proof. The morphism $L\uparrow_H^G \to M\uparrow_H^G$ evaluated on a G-set X is by definition the evaluation of $L \to M$ on the H-set $X\downarrow_H^G$, which is a monomorphism (resp. epimorphism) by assumption. More generally the exactness of induction follows in the same manner. It is also easy to give a proof using the elementary approach to induction. □

5. Inflation.

Let N be a normal subgroup of G with quotient $Q = G/N$. Given a Mackey functor M for Q we define a Mackey functor $\mathrm{Inf}_Q^G M$ called the *inflation* of M from Q to G, as follows:

$$\mathrm{Inf}_Q^G M(H) = \begin{cases} 0 & \text{if } H \not\geq N \\ M(H/N) & \text{if } H \geq N. \end{cases}$$

The restriction and transfer morphisms r_H^K, t_H^K are zero unless $N \subseteq H \subseteq K$ in which case they are the mappings $r_{H/N}^{K/N}, t_{H/N}^{K/N}$ for M. Similarly conjugations are inherited from M.

We now describe the adjoints of the inflation functor. Given a Mackey functor L on G we define Mackey functors L^+ and L^- on Q as follows.

$$L^+(K/N) = L(K) / \sum_{\substack{J \leq K \\ J \not\geq N}} t_J^K L(J)$$

$$L^-(K/N) = \bigcap_{\substack{J \leq K \\ J \not\geq N}} \mathrm{Ker}\, r_J^K$$

Restriction, transfer and conjugation mappings come from those for L, for which we rely on the fact that those mappings for L preserve the sum of images of transfers and the intersection of kernels of restrictions.

(5.1) PROPOSITION. $^+$ *is left adjoint to* Inf_Q^G, *and* $^-$ *is right adjoint to* Inf_Q^G.

Proof. Let L be a Mackey functor for G and M a Mackey functor for Q. Any morphism $\alpha : L \to \mathrm{Inf}_Q^G M$ is necessarily zero on $L(J)$ with $J \not\geq N$, and hence must vanish on $\sum_{\substack{J \leq K \\ J \not\geq N}} t_J^K L(J)$ for each subgroup K. Thus α induces a morphism $L^+ \to M$. Conversely, for any morphism $\beta : L^+ \to M$ we construct a morphism $L \to \mathrm{Inf}_Q^G M$ by defining it to be zero on $L(J)$ when $J \not\geq N$, and defining it to be the composite

$$L(K) \to L(K) / \sum_{\substack{J \leq K \\ J \not\geq N}} t_J^K L(J) = L^+(K/N) \to M(K/N) = \mathrm{Inf}_Q^G M(K)$$

when $K \geq N$. The fact that this composite vanishes on $\sum_{\substack{J \leq K \\ J \not\geq N}} t_J^K L(J)$ ensures that it is genuinely a morphism of Mackey functors. By these constructions we establish a natural bijection $\mathrm{Hom}(L, \mathrm{Inf}_Q^G M) \leftrightarrow \mathrm{Hom}(L^+, M)$ which shows that $^+$ is left adjoint to Inf_Q^G. The proof that $^-$ is right adjoint to Inf_Q^G is similar. $\square$

The fact that inflated functors vanish on subgroups not containing the normal subgroup is important for the following result which will turn out to be crucial in the classification of simple functors.

(5.2) PROPOSITION. *Let H be a subgroup of G and L a Mackey functor for the group $\overline{N}_G(H) = N_G(H)/H$. Then the Mackey functor $M = (\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)} L) \uparrow_{N_G(H)}^{G}$ for the group G satisfies $M(H) = L(1)$.*

Proof. Proposition 4.3 gives

$$M(K) = \bigoplus_{g \in [K \backslash G / N_G(H)]} (\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)} L)(N_G(H) \cap K^g).$$

The only non-zero terms have $N_G(H) \cap K^g \geq H$, in other words $K^g \geq H$. When $K = H$ this condition becomes $g \in N_G(H)$, but there is only one double coset representative with this property, namely 1, so $M(H) = (\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)} L)(H) = L(1)$. □

6. Fixed point and fixed quotient Mackey functors

Let V be a (left) kG-module. We define the *fixed point Mackey functor* FP_V to be the Mackey functor such that $FP_V(H) = V^H$, where V^H denotes the set of all H-fixed points in V. The *fixed quotient Mackey functor* FQ_V is the Mackey functor such that $FQ_V(H) = V_H$, where V_H denotes the largest quotient of V on which H acts trivially. For FP_V, the restriction $r_K^H : V^H \to V^K$ is the inclusion of fixed points and the transfer $t_K^H : V^K \to V^H$ is the *relative trace map* sending $x \in V^K$ to $\sum_{h \in [H/K]} h \cdot x$. Similarly for FQ_V, the restriction $r_K^H : V_H \to V_K$ is induced by multiplication by $\sum_{h \in [K \backslash H]} h$ and the transfer $t_K^H : V_K \to V_H$ is the surjection of fixed quotients. For both Mackey functors, the conjugation c_g is multiplication by g. Evidently $FP_V(1) = V = FQ_V(1)$.

It is clear that a homomorphism $V \to W$ of kG-modules induces morphisms of Mackey functors $FP_V \to FP_W$ and $FQ_V \to FQ_W$. Thus FP and FQ define two (fully faithful) functors from the category kG-mod of left kG-modules to the category $\mathrm{Mack}(G)$.

There is also an obvious forgetful functor $E : \mathrm{Mack}(G) \to kG$-mod mapping a Mackey functor M to its evaluation $M(1)$ at the trivial subgroup 1 of G.

(6.1) PROPOSITION. *The left adjoint of E is FQ, and the right adjoint of E is FP. Explicitly let M be a Mackey functor for G and V a kG-module.*

(i) Given any morphism of kG-modules $\alpha : M(1) \to V$, there exists a unique morphism of Mackey functors $M \to FP_V$ which extends α.

(ii) Given any morphism of kG-modules $\beta : V \to M(1)$, there exists a unique morphism of Mackey functors $FQ_V \to M$ which extends β.

Proof. (i) We define $\alpha_H : M(H) \to V^H = FP_V(H)$ as the composition $\alpha_H = \alpha r_1^H$. In particular $\alpha_1 = \alpha$. The image of αr_1^H is contained in V^H, because for $h \in H$, since h acts trivially on $M(H)$ and commutes with α, we have $c_h\, \alpha\, r_1^H = \alpha\, r_{^h 1}^{^h H}\, c_h = \alpha\, r_1^H$. Furthermore, commutativity of the following diagram implies that our choice of α_H is unique.

$$\begin{array}{ccc} M(H) & \xrightarrow{\alpha_H} & V^H \\ {\scriptstyle r_1^H}\downarrow & & \downarrow{\scriptstyle \text{inclusion}} \\ M(1) & \xrightarrow[\alpha]{} & V \end{array}$$

We verify the two equations

$$\alpha_H\, t_K^H = t_K^H\, \alpha_K$$
$$\alpha_K\, r_K^H = r_K^H\, \alpha_H\ .$$

Substituting α_H into the first equation and using Mackey's axiom we obtain

$$\alpha_H\, t_K^H = \alpha\, r_1^H\, t_K^H = \alpha \sum_{h\in[H/K]} c_h\, r_1^K = \sum_{h\in[H/K]} h\cdot \alpha\, r_1^K = t_K^H\, \alpha\, r_1^K = t_K^H\, \alpha_K\ .$$

Treating the second equation similarly, we obtain

$$\alpha_K\, r_K^H = \alpha\, r_1^K\, r_K^H = \alpha\, r_1^H = \alpha_H = r_K^H\, \alpha_H$$

since r_K^H for the Mackey functor FP_V is inclusion $V^H \subseteq V^K$. This proves that the proposed morphism of Mackey functors commutes with transfers and restrictions. It commutes with conjugations because both mappings in its definition do.

(ii) We define $\beta_H : V_H \to M(H)$ to be the unique map making the following diagram commutative.

$$\begin{array}{ccc} V_H & \xrightarrow{\beta_H} & M(H) \\ {\scriptstyle \text{quotient}}\uparrow & & \uparrow{\scriptstyle t_1^H} \\ V & \xrightarrow{\beta} & M(1) \end{array}$$

Then one verifies that β is a morphism of Mackey functors in a way which is dual to that of part (i). $\square$

(6.2) Remark. There is a duality for Mackey functors which interchanges FP and FQ, so that the arguments for FQ are indeed dual of those for FP in a more precise way. In more detail, for any Mackey functor M, one defines the dual M^* of M by $M^*(H) = M(H)^*$, the dual of the vector space $M(H)$. The restriction maps for M^* are the dual maps of the transfers for M, while the transfer maps for M^* are the dual maps of the restrictions for M. Morover the conjugation by g in M^* is the dual of the conjugation by g^{-1} in M. It is not difficult to prove that $(FP_V)^* \cong FQ_{V^*}$ where V^* is the dual (contragredient) representation of V.

7. Simple Mackey functors with trivial minimal subgroup

In this section, we classify the simple Mackey functors $S = S_{1,V}$ with trivial minimal subgroup (that is, such that $S(1) \neq 0$). They are parameterised by pairs $(1, V)$ where 1 is the trivial subgroup and V a simple kG-module. We first describe $S_{1,V}$ explicitly.

(7.1) LEMMA. *Let V be a simple kG-module.*

(i) The fixed point functor FP_V has a unique minimal subfunctor $S_{1,V}$, identified by $S_{1,V}(H) = \mathrm{Im}(t_1^H : V \to V^H)$. In particular, $S_{1,V}$ is a simple Mackey functor.

(ii) The fixed quotient functor FQ_V has a unique simple quotient functor $S_{1,V}$, identified by $S_{1,V}(H) = V_H/\mathrm{Ker}(r_1^H : V_H \to V_1)$.

(iii) The simple Mackey functors $S_{1,V}$ defined in (i) and (ii) are isomorphic.

Proof. (i) Let $0 \neq M$ be a subfunctor of FP_V. For some subgroup $K \leq G$, $M(K) \neq 0$. Since $r_1^K : FP_V(K) \to FP_V(1)$ is the inclusion of fixed points we have $M(1) \neq 0$. But then $M(1) = V$ since $M(1)$ is a kG-submodule of the simple module V. Therefore M contains the subfunctor $< FP_V(1) >$ generated by $FP_V(1) = V$. But this latter functor is precisely $S_{1,V}$ (see Proposition 2.1 applied with $\mathcal{X} = \{1\}$).

(ii) The proof is similar to (i), using a dual argument.

(iii) The adjointness property of either FP_V or FQ_V (Proposition 6.1) applied to the situation $FQ_V(1) = V = FP_V(1)$ yields a morphism $\alpha : FQ_V \to FP_V$. Because the transfer maps t_K^H for FQ_V are always surjective, the same holds for the image of α, and hence the value of the image at a subgroup K is $t_1^K V = S_{1,V}(K)$. Thus the image of α is $S_{1,V}$. Since FQ_V has a unique simple quotient, it is mapped isomorphically onto $S_{1,V}$. □

(7.2) THEOREM. *The simple Mackey functors with trivial minimal subgroup are precisely the Mackey functors $S_{1,V}$ with V a simple kG-module. Moreover $S_{1,V} \cong S_{1,W}$ if and only if $V \cong W$.*

Proof. We have already seen in Lemma 7.1 that the functors $S_{1,V}$ are simple. On the other hand if M is a simple Mackey functor with $M(1) \neq 0$ then $M(1) = V$ is a simple kG-module by Proposition 2.3. Moreover the identity morphism $M(1) = V$ extends uniquely to a morphism $\alpha : M \to FP_V$ by Proposition 6.1. By simplicity of M, the kernel of α (which is obviously a subfunctor of M) is zero and therefore α gives an isomorphism from M to the unique simple subfunctor $S_{1,V}$ of FP_V. The second claim is clear since on the one hand V determines $S_{1,V}$ as the unique minimal subfunctor of FP_V and on the other hand $S_{1,V}$ determines V as its evaluation at 1. □

8. The classification of simple Mackey functors

In this section we classify the simple Mackey functors $S = S_{H,V}$. As mentioned in Section 2, they are parameterised by pairs (H, V) where H is a minimal subgroup of S (defined up to conjugacy) and $V = S(H)$ is a simple $k\overline{N}_G(H)$-module. First we describe $S_{H,V}$ explicitly. We write $S^G_{H,V} = S_{H,V}$ in order to emphasize that it is a Mackey functor for G.

Let H be a subgroup of G and let V be a simple $k\overline{N}_G(H)$-module. Let $S^{\overline{N}_G(H)}_{1,V}$ be the simple Mackey functor for the group $\overline{N}_G(H)$ constructed in the previous section. Thus $S^{\overline{N}_G(H)}_{1,V}$ is the unique minimal subfunctor of the fixed point functor FP_V.

(8.1) LEMMA. *Let H be a subgroup of G and let V be a simple $k\overline{N}_G(H)$-module.*

(i) $M = (\mathrm{Inf}^{N_G(H)}_{\overline{N}_G(H)} FP_V) \uparrow^G_{N_G(H)}$ *has a unique minimal subfunctor* $S^G_{H,V}$*, generated by* $M(H) = V$*. This minimal subfunctor is isomorphic to* $(\mathrm{Inf}^{N_G(H)}_{\overline{N}_G(H)} S^{\overline{N}_G(H)}_{1,V}) \uparrow^G_{N_G(H)}$.

(ii) $(\mathrm{Inf}^{N_G(H)}_{\overline{N}_G(H)} FQ_V) \uparrow^G_{N_G(H)}$ *has a unique maximal subfunctor. The quotient is isomorphic to* $S^G_{H,V} \cong (\mathrm{Inf}^{N_G(H)}_{\overline{N}_G(H)} S^{\overline{N}_G(H)}_{1,V}) \uparrow^G_{N_G(H)}$.

Proof. (i) Let T be any subfunctor of M. We show that $T(H) = M(H)$, so that $T \supseteq <M(H)>$. From this it will follow that the Mackey functor $< M(H) >$ generated by $M(H)$ is the unique minimal subfunctor of M. By Proposition 5.2, $M(H) = FP_V(1) = V$ is a simple $k\overline{N}_G(H)$-module. Since $T(H)$ is a $k\overline{N}_G(H)$-submodule of $M(H)$, it suffices to prove that $T(H) \neq 0$. For simplicity, we write Inf instead of $\mathrm{Inf}^{N_G(H)}_{\overline{N}_G(H)}$.

By adjointness properties (Propositions 4.2, 5.1 and 6.1) we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Mack}(G)}(T, M) &\cong \mathrm{Hom}_{\mathrm{Mack}(N_G(H))}(T\downarrow^G_{N_G(H)}, \mathrm{Inf}\, FP_V)\\
&\cong \mathrm{Hom}_{\mathrm{Mack}(\overline{N}_G(H))}\big((T\downarrow^G_{N_G(H)})^+, FP_V\big)\\
&\cong \mathrm{Hom}_{k\overline{N}_G(H)}\big((T\downarrow^G_{N_G(H)})^+(1), V\big)\\
&= \mathrm{Hom}_{k\overline{N}_G(H)}\big((T\downarrow^G_{N_G(H)})(H/\sum_{\substack{J\leq H\\ J\not\supseteq H}} t^H_J T(J)), V\big)\\
&= \mathrm{Hom}_{k\overline{N}_G(H)}(T(H), V)
\end{aligned}$$

the last equality coming from the fact that M and hence T vanish on proper subgroups of H. The inclusion of T in M being a non-zero morphism of the left-hand side, there is a non-zero homomorphism from $T(H)$ to V and this forces $T(H) \neq 0$.

We now turn to the second claim in part (i). Since both inflation and induction are exact functors (cf Proposition 4.4), we may regard $(\mathrm{Inf}\, S^{\overline{N}_G(H)}_{1,V}) \uparrow^G_{N_G(H)}$ as a subfunctor of the functor

$M = (\mathrm{Inf}\, FP_V) \uparrow_{N_G(H)}^G$. Thus it suffices to show that it is generated by $M(H)$. By Proposition 4.3, for any subgroup K we have the formula

$$(8.2) \qquad (\mathrm{Inf}\, S_{1,V}^{\overline{N}_G(H)}) \uparrow_{N_G(H)}^G (K) = \bigoplus_{g \in [K \backslash G / N_G(H)]} (\mathrm{Inf}\, S_{1,V}^{\overline{N}_G(H)})(N_G(H) \cap K^g).$$

The only non-zero summands satisfy $K^g \supseteq H$. Fix such a g and consider the value of the functor at ${}^gH$. By a similar computation (or by Proposition 5.2) we have

$$\begin{aligned}(\mathrm{Inf}\, S_{1,V}^{\overline{N}_G(H)}) \uparrow_{N_G(H)}^G ({}^gH) &= (\mathrm{Inf}\, S_{1,V}^{\overline{N}_G(H)})(N_G(H) \cap ({}^gH)^g) = (\mathrm{Inf}\, S_{1,V}^{\overline{N}_G(H)})(H) \\ &= V = M(H).\end{aligned}$$

Proposition 4.3 tells us that the transfer $t_{{}^gH}^K$ is equal to $t_H^{N_G(H) \cap K^g}$ having codomain the summand corresponding to g in (8.2). This map is surjective since $t_H^{N_G(H) \cap K^g}$ in the Mackey functor $\mathrm{Inf}\, S_{1,V}^{\overline{N}_G(H)}$ is the transfer $t_1^{\overline{N}_{K^g}(H)}$ in the Mackey functor $S_{1,V}^{\overline{N}_G(H)}$ which is surjective by Lemma 7.1.

It follows from the above analysis that the sum of the transfer maps $\sum_{\substack{g \in G \\ {}^gH \subseteq K}} t_{{}^gH}^K$ is surjective onto $(\mathrm{Inf}\, S_{1,V}^{\overline{N}_G(H)}) \uparrow_{N_G(H)}^G (K)$, which completes the proof that the subfunctor $(\mathrm{Inf}\, S_{1,V}^{\overline{N}_G(H)}) \uparrow_{N_G(H)}^G$ is generated by $M(H) = V$.

(ii) is proved by a dual argument. □

Now we can prove the main result of this paper. Recall from Section 2 that Ω is the set of pairs (H, V) where H is a subgroup of G and V a simple $k\overline{N}_G(H)$-module (up to isomorphism). Also G acts by conjugation on Ω and we have a map

$$\Phi : \mathrm{Irr}_k(\mathrm{Mack}(G)) \longrightarrow \Omega/G$$

which associates to a simple Mackey functor S the G-conjugacy class of (H, V) where H is a minimal subgroup of S and $V = S(H)$.

(8.3) THEOREM. *The map Φ is a bijection. Its inverse maps a pair (H, V) to the simple functor $S_{H,V}$ constructed in Lemma 8.1. In other words the Mackey functors $S_{H,V}$ form a complete list of non-isomorphic simple Mackey functors (for $(H, V) \in \Omega/G$).*

Proof. By Lemma 8.1, the Mackey functors $S_{H,V} = S_{H,V}^G = (\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)}\, S_{1,V}^{\overline{N}_G(H)}) \uparrow_{N_G(H)}^G$ are simple. Also by construction H is a minimal subgroup of $S_{H,V}$ and $V = S_{H,V}(H)$. Therefore we have constructed a right inverse of the map Φ and it suffices to prove that any simple Mackey functor S is isomorphic to $S_{H,V}$ for H a minimal subgroup of S and $V = S(H)$. For simplicity, we write again Inf instead of $\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)}$. Since $S_{H,V}$ is the unique minimal subfunctor of $(\mathrm{Inf}\, FP_V) \uparrow_{N_G(H)}^G$, we only need to establish the existence of a non-zero morphism $\alpha : S \to (\mathrm{Inf}\, FP_V) \uparrow_{N_G(H)}^G$. Indeed by simplicity of S, the kernel of α (which is obviously a subfunctor of S) is zero and therefore α

gives an isomorphism from S to the unique simple subfunctor $S_{H,V}$ of $(\mathrm{Inf}\, FP_V) \uparrow_{N_G(H)}^G$. The existence of α follows from a sequence of adjunctions which we have already encountered in the proof of Lemma 8.1.

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Mack}(G)}\,(S\,,\,(\mathrm{Inf}\, FP_V)\uparrow_{N_G(H)}^G) &\cong \mathrm{Hom}_{\mathrm{Mack}(N_G(H))}\,(S\downarrow_{N_G(H)}^G\,,\,\mathrm{Inf}\, FP_V) \\
&\cong \mathrm{Hom}_{\mathrm{Mack}(\overline{N}_G(H))}\,\big((S\downarrow_{N_G(H)}^G)^+\,,\,FP_V\big) \\
&\cong \mathrm{Hom}_{k\overline{N}_G(H)}\,\big((S\downarrow_{N_G(H)}^G)^+(1)\,,\,V\big) \\
&= \mathrm{Hom}_{k\overline{N}_G(H)}\,\big((S\downarrow_{N_G(H)}^G)\,(H/\sum_{\substack{J\leq H\\ J\ngeq H}} t_J^H\, S(J))\,,\,V\big) \\
&= \mathrm{Hom}_{k\overline{N}_G(H)}\,(S(H),V)\,.
\end{aligned}$$

The last homomorphism group is non-zero because $S(H)=V$ and so the proof is complete. □

9. Invertible group order

In this section we assume that our base field k has characteristic zero or prime to the order of the group G. In other words we assume that $|G|$ is invertible in k. Then we have the following semi-simplicity result, analogous to Maschke's theorem.

(9.1) THEOREM. *Assume that $|G|$ is invertible in k. Then every Mackey functor for G over k is a direct sum of simple Mackey functors.*

As in the case of modules, the statement is equivalent to the claim that every subfunctor of M is a direct summand of M (as a Mackey functor), or also that every short exact sequence of Mackey functors splits.

(9.2) LEMMA. *Assume that $|G|$ is invertible in k. Then the simple Mackey functor $S_{H,V}$ is equal to $(\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)}\, FP_V)\uparrow_{N_G(H)}^G$.*

Proof. By Lemma 8.1, we know that $S_{H,V} = (\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)}\, S_{1,V}^{\overline{N}_G(H)})\uparrow_{N_G(H)}^G$ where $S_{1,V}^{\overline{N}_G(H)}$ is the unique simple subfunctor of FP_V for the group $\overline{N}_G(H)$. Thus it suffices to prove that $S_{1,V}^{\overline{N}_G(H)} = FP_V$ and so we may assume that $H=1$. By Lemma 7.1, we have

$$S_{1,V}(K) = \mathrm{Im}(t_1^K : V \to V^K) \subseteq V^K = FP_V(K)$$

for every subgroup K. Therefore it suffices to prove that t_1^K is surjective onto V^K. But this is clear since for $x \in V^K$, we have $x = |K|^{-1}\, t_1^K(x)$. □

First proof of (9.1). This proof is based on one of the main results of [T 1]. For a Mackey functor M and for each subgroup H, we define

$$\overline{M}(H) = M(H) / \sum_{K<H} t_K^H(M(K))$$

and we let $\mathcal{P}(M)$ denote the set of subgroups H such that $\overline{M}(H) \neq 0$ (called *primordial* subgroups in [T 1]). The group G acts by conjugation on $\mathcal{P}(M)$. Now recall that the *twin functor* TM of M is defined by

$$TM(H) = \Big(\bigoplus_{K \leq H} \overline{M}(K) \Big)^H$$

where the exponent H denotes the set of H-fixed points in the direct sum with respect to the conjugation action which is part of the definition of a Mackey functor. If we let $R(M) = \bigoplus_{K \leq G} \overline{M}(K)$ with its natural conjugation action, then $TM(H) \subseteq R(M)^H$, by taking zero for the components indexed by a subgroup K not contained in H. It is easy to see that TM is a subfunctor of the fixed-point functor $FP_{R(M)}$.

The result of [T 1] which we need asserts that when $|G|$ is invertible, every Mackey functor M is isomorphic to its twin functor TM. This is Corollary 4.4 in [T 1] for Mackey functors endowed with a multiplicative structure (called G-functors in [T 1]), but as mentioned in Theorem 12.3 of that paper, the result also holds for arbitrary Mackey functors (called module G-functors in [T 1]). Thus it sufices to prove our theorem for TM instead of M.

Now we claim that

$$TM \cong \bigoplus_{H \in [\mathcal{P}(M)/G]} \big(\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)} (FP_{\overline{M}(H)})\big) \uparrow_{N_G(H)}^G \quad . \tag{9.3}$$

The proof is exactly the same as that of Proposition 11.4 in [T 2]. Let K be a subgroup of G. Then

$$TM(K) = \Big(\bigoplus_{P \leq K} \overline{M}(P) \Big)^K = \bigoplus_{P \in [\mathcal{P}(M)/G]} \Big(\bigoplus_{\substack{g \in [G/N_G(P)] \\ {}^gP \leq K}} \overline{M}({}^gP) \Big)^K .$$

It is clear that this gives a decomposition of TM as a direct sum, so we now fix some $P \in \mathcal{P}(M)$. We have

$$\begin{aligned}
\Big(\bigoplus_{\substack{g \in [G/N_G(P)] \\ {}^gP \leq K}} \overline{M}({}^gP) \Big)^K &\cong \bigoplus_{\substack{g \in [K\backslash G/N_G(P)] \\ {}^gP \leq K}} \overline{M}({}^gP)^{N_K({}^gP)} \\
&\cong \bigoplus_{\substack{g \in [K\backslash G/N_G(P)] \\ P \leq K^g}} \overline{M}(P)^{N_{K^g}(P)} \\
&= \bigoplus_{g \in [K\backslash G/N_G(P)]} \big(\mathrm{Inf}_{\overline{N}_G(P)}^{N_G(P)} (FP_{\overline{M}(P)})\big)(N_G(P) \cap K^g) \\
&= \big(\mathrm{Inf}_{\overline{N}_G(P)}^{N_G(P)} (FP_{\overline{M}(P)})\big) \uparrow_{N_G(P)}^G (K) \, .
\end{aligned}$$

The proof that these isomorphisms commute with restriction, transfer and conjugation is left to the reader.

We have now to prove that each summand in (9.3) is a direct sum of simple Mackey functors. By Maschke's theorem for the group $\overline{N}_G(H)$, the $k\overline{N}_G(H)$-module $\overline{M}(H)$ is a direct sum of simple submodules V_i. Therefore $FP_{\overline{M}(H)} \cong \bigoplus_i FP_{V_i}$ and since both inflation and induction commute with direct sums, we see that the summand indexed by H in (9.3) is a direct sum of Mackey functors of the form $\left(\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)}(FP_V)\right)\uparrow_{N_G(H)}^G$ where V is a simple $k\overline{N}_G(H)$-module. By Lemma 9.2, every such Mackey functor is simple. □

Second proof of (9.1). This proof is based on the following well-known result which is a consequence of the analysis of the action of the Burnside ring functor on an arbitrary Mackey functor M (using the semi-simplicity of the Burnside algebra over a field in which $|G|$ is invertible).

(9.4) LEMMA. *Assume $|G|$ is invertible in k and let M be a Mackey functor for G. Let $\mathcal{X}$ be a family of subgroups of G closed under subconjugation. Then*

$$M = \mathrm{Im}\, t_{\mathcal{X}} \oplus \mathrm{Ker}\, r_{\mathcal{X}}\,.$$

As in Section 3, we write $\mathrm{Im}\, t_{\mathcal{X}}$ and $\mathrm{Ker}\, r_{\mathcal{X}}$ for the subfunctors of M defined by

$$(\mathrm{Im}\, t_{\mathcal{X}})(K) = \sum_{\substack{X\in\mathcal{X}\\ X\leq K}} \mathrm{Im}\, t_X^K \quad \text{and} \quad (\mathrm{Ker}\, r_{\mathcal{X}})(K) = \bigcap_{\substack{X\in\mathcal{X}\\ X\leq K}} \mathrm{Ker}\, r_X^K\,.$$

The lemma is a very special case of Dress's induction theorem [Dr, Theorem 4]. For a direct proof, see [T 1, Proposition 8.5] together with the remarks made in Section 12 of that paper.

For convenience, we shall say that M is $\mathcal{X}$*-reduced* if either $M = \mathrm{Im}\, t_{\mathcal{X}}$ (and so $\mathrm{Ker}\, r_{\mathcal{X}} = 0$) or $M = \mathrm{Ker}\, r_{\mathcal{X}}$ (and so $\mathrm{Im}\, t_{\mathcal{X}} = 0$). Also we shall call M *reduced* if it is $\mathcal{X}$-reduced for every subconjugacy closed family of subgroups $\mathcal{X}$.

(9.5) LEMMA. *Assume $|G|$ is invertible in k and let M be a Mackey functor for G. Then M decomposes as a direct sum of reduced subfunctors.*

Proof. From the definition of $\mathrm{Im}\, t_{\mathcal{X}}$ and $\mathrm{Ker}\, r_{\mathcal{X}}$, it is immediate that any direct summand of an $\mathcal{X}$-reduced Mackey functor is again $\mathcal{X}$-reduced. Order arbitrarily the set of all subconjugacy closed families $\mathcal{X}$, say $\{\mathcal{X}_1, \ldots, \mathcal{X}_n\}$. By induction, consider a decomposition of M into direct summands which are $\mathcal{X}_i$-reduced for all $i \leq k-1$ and let N be one of the summands. By Lemma 9.4, $N = \mathrm{Im}\, t_{\mathcal{X}} \oplus \mathrm{Ker}\, r_{\mathcal{X}}$. Both $\mathrm{Im}\, t_{\mathcal{X}}$ and $\mathrm{Ker}\, r_{\mathcal{X}}$ are now $\mathcal{X}_i$-reduced for all $i \leq k$. The result follows by induction. □

Now we can finish the proof of (9.1). Let M be a non-zero Mackey functor over k. By Lemma 9.5, we can assume that M is reduced. For a subgroup H of G, let $\mathcal{X}(H)$ be the set of conjugates of subgroups of H. Since $M \neq 0$, we have $M = \operatorname{Im} t_{\mathcal{X}(G)}$, and therefore there exists a minimal subgroup H such that $M = \operatorname{Im} t_{\mathcal{X}(H)}$. Then for any $K < H$, we have $\operatorname{Im} t_{\mathcal{X}(K)} = 0$ and $M = \operatorname{Ker} r_{\mathcal{X}(K)}$ because M is reduced. In particular $M(K) = 0$ and it follows that

$$(9.6) \qquad M\downarrow_{\mathcal{X}(H)}(K) = \begin{cases} M(K) & \text{if } K =_G H\,, \\ 0 & \text{if } K <_G H\,. \end{cases}$$

Choose a decomposition of $M(H)$ as a direct sum of simple $k\overline{N}_G(H)$-modules (which exists by Maschke's theorem). This induces a corresponding decomposition of $M({}^gH)$ for all $g \in G$, and hence a decomposition of $M\downarrow_{\mathcal{X}(H)}$ as a Mackey functor defined on $\mathcal{X}(H)$ (because by (9.6), all restriction and transfer maps are zero). Since M is $\mathcal{X}(H)$-reduced, Proposition 3.2 applies and therefore M decomposes as a direct sum of subfunctors N_i such that $N_i(H)$ is a simple $k\overline{N}_G(H)$-module. Each N_i is again $\mathcal{X}(H)$-reduced. Thus the three conditions of Theorem 3.1 are satisfied, proving that N_i is a simple Mackey functor. □

The primitive idempotents of the Burnside algebra of G (over a field k in which $|G|$ is invertible) are indexed by the conjugacy classes of subgroups of G. If e_H is the idempotent corresponding to the conjugacy class of H, then for an arbitrary Mackey functor M over k, we have a subfunctor M_H of M defined by $M_H(K) = r^G_K(e_H) \cdot M(K)$ where the dot denotes the natural action of the Burnside algebra of K on $M(K)$, see [t D, Proposition 6.2.3] or [T 1, Section 12]. It is not difficult to show that M_H is reduced and that $M = \bigoplus_H M_H$ is the unique decomposition of M into reduced summands. Moreover M_H vanishes on subgroups not containing a conjugate of H (but not on H, as in (9.6)) and decomposes as a direct sum of simple functors with minimal subgroup H. This provides a more explicit way of decomposing an arbitrary Mackey functor.

Both proofs of (9.1) rely on two ingredients. The first is the semi-simplicity of the Burnside algebra over k. This result implies (9.4) (second proof), while in the first proof, the fact that any Mackey functor is isomorphic to its twin functor is a result of a very similar nature (which actually implies the semi-simplicity of the Burnside algebra, see [T 1, Section 6]). The second ingredient is the semi-simplicity of the group algebra $k\overline{N}_G(H)$ which in both proofs allows to decompose a functor with minimal subgroup H.

The first proof has the merit of making clear which simple functors $S_{H,V}$ appear in a decomposition of M; namely V must be one of the direct summand of the semi-simple $k\overline{N}_G(H)$-module $\overline{M}(H) = M(H)/\sum_{K<H} t^H_K(M(K))$. The second proof has the advantage of being independent of the classification of simple functors, since it only uses the simplicity criterion of Theorem 3.1.

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