COMBINATORIAL RESTRICTIONS ON THE TREE CLASS OF THE AUSLANDER-REITEN QUIVER OF A TRIANGULATED CATEGORY

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Abstract. We show that if a connected, Hom-finite, Krull-Schmidt triangulated category has an Auslander-Reiten quiver component with Dynkin tree class then the category has Auslander-Reiten triangles and that component is the entire quiver. This is an analogue for triangulated categories of a theorem of Auslander, and extends a previous result of Scherotzke. We also show that if there is a quiver component with extended Dynkin tree class, then other components must also have extended Dynkin class or one of a small set of infinite trees, provided there is a non-zero homomorphism between the components. The proofs use the theory of additive functions.

1. Main results

Let k be a field and let $\mathcal{C}$ be a k-linear triangulated category which is Hom-finite, Krull-Schmidt and connected. The Auslander-Reiten quiver of $\mathcal{C}$ is the graph whose vertices are the indecomposable objects of $\mathcal{C}$ (up to isomorphism) and where we draw an arrow from X to Y, labelled with certain multiplicity information, if there is an irreducible morphism from X to Y. We will be concerned with parts of the Auslander-Reiten quiver where Auslander-Reiten triangles exist. If $U \to V \to W \to U[1]$ is an Auslander-Reiten triangle we write $U = \tau W$ and $W = \tau^{-1}U$ to define the *Auslander-Reiten translate* τ.

By a *stable component* Γ of the Auslander-Reiten quiver we mean a subgraph with the properties:

(1) for every indecomposable object $M \in \Gamma$, for every $n \in \mathbb{Z}$, $\tau^n M$ also lies in Γ and,
(2) every irreducible morphism beginning or ending at M lies in Γ.

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A stable component Γ has the form $\mathbb{Z}T/G$ where T is a 'valued' tree called the *tree class* of Γ, and G is a group of automorphisms of $\mathbb{Z}T$. This result originates with Riedtmann [6] under the assumption that k be algebraically closed. It was observed in [5, page 288] that her theory works for arbitrary fields, and an extension to triangulated categories was made in [12]. In this definition we do not suppose that Γ is closed under the shift operation of $\mathcal{C}$, and we note that a stable component is actually a component of the Auslander-Reiten quiver, not a subset of a component with some objects removed. When Auslander-Reiten triangles exist everywhere in $\mathcal{C}$ all components are stable, but we wish to consider situations where they do not always exist, such as bounded derived categories of finite dimensional algebras which are not necessarily of finite global dimension.

We say that $\mathcal{C}$ is *locally finite* if it is Hom-finite and for each indecomposable object X there are only finitely many isomorphism classes of indecomposable objects Y for which $\mathrm{Hom}_{\mathcal{C}}(X, Y) \neq 0$ or $\mathrm{Hom}_{\mathcal{C}}(Y, X) \neq 0$. This is a strong condition which implies that Auslander-Reiten triangles do always exist, there is a single Auslander-Reiten quiver component, and it has Dynkin tree class. This was proved by Xiao and Zhu [11, 12]. Under the hypothesis of local finiteness, such categories were (partially) classified by Amiot [1].

Our first main result is a converse to this.

Theorem 1.1. *Let $\mathcal{C}$ be a Hom-finite, Krull-Schmidt, connected triangulated category, let Γ be a stable component of the Auslander-Reiten quiver of $\mathcal{C}$ and suppose that Γ has Dynkin tree class. Then Γ contains every indecomposable object of $\mathcal{C}$. Furthermore $\mathcal{C}$ is locally finite. It follows that $\mathcal{C}$ has Auslander-Reiten triangles and has only finitely many shift-classes of indecomposable objects.*

In the particular case of bounded derived categories of algebras our result was proved already by Scherotzke [7] using quite different methods: she related irreducible morphisms in the derived category to irreducible morphisms in the category of complexes and chain maps. Our own approach uses only the theory of additive functions. It applies more generally and has the merit of being brief.

The theorem is a version for triangulated categories of the theorem of Auslander for modules over an indecomposable finite dimensional algebra, which states that if the Auslander-Reiten quiver of the module category has a finite component then it is the entire quiver. For a triangulated category the condition that there is a finite stable component Γ is strong and does imply that the component must have Dynkin tree class. We state this and give the short proof.

Corollary 1.2. *Let $\mathcal{C}$ be a Hom-finite, Krull-Schmidt, connected triangulated category with a finite stable component Γ of its Auslander-Reiten quiver. Then the tree class of Γ is a Dynkin diagram and Γ contains every indecomposable object of $\mathcal{C}$.*

Proof. We may use the argument of [12, Theorem 2.3.5]: the function

$$f(X) := \sum_{M\in\Gamma} \dim \operatorname{Hom}_{\mathcal{C}}(M, X)$$

is subadditive and periodic on Γ, and this forces Γ to have Dynkin tree class by [5]. The result now follows by Theorem 1.1. □

We also have a theorem about Auslander-Reiten quiver components which have extended Dynkin tree class.

Theorem 1.3. *Let $\mathcal{C}$ be a Hom-finite, Krull-Schmidt triangulated category with stable Auslander-Reiten quiver components Γ_1 and Γ_2. Suppose there are objects $X \in \Gamma_1$ and $Y \in \Gamma_2$ so that either $\operatorname{Hom}(X, Y) \neq 0$ or $\operatorname{Hom}(Y, X) \neq 0$. If Γ_1 has an extended Dynkin diagram as its tree class then the tree class of Γ_2 is either an extended Dynkin diagram or one of the trees A_∞, B_∞, C_∞, D_∞ or A_∞^∞.*

The above trees are displayed in [2, 5, 9], for instance. Although it is remarkable that the shape of one quiver component influences the shape of other components in this way, the theorem still allows for many possibilities and we wonder if they can all occur in triangulated categories. We ask the following:

Question 1.4. With the hypotheses and notation of Theorem 1.3, is it possible to have stable Auslander-Reiten quiver components Γ_1 and Γ_2 with $\operatorname{Hom}(X, Y) \neq 0$ for some $X \in \Gamma_1$ and $Y \in \Gamma_2$, so that Γ_1 and Γ_2 have different (finite) extended Dynkin diagrams as their tree classes? If Γ_1 has extended Dynkin tree class, is it even the case that Γ_2 must either have the same tree class as Γ_1 or else have tree class A_∞?

We mention the definitions of some of the standard terms we have been using. We say that $\mathcal{C}$ is *Hom-finite* if for every pair of indecomposable objects X and Y in $\mathcal{C}$, $\dim_k \operatorname{Hom}_{\mathcal{C}}(X, Y)$ is finite; we say that $\mathcal{C}$ is *Krull-Schmidt* if every object in $\mathcal{C}$ is a finite direct sum of indecomposable objects each of which has a local endomorphism ring; and we say that $\mathcal{C}$ is *connected* if it is not possible to partition the indecomposable objects of $\mathcal{C}$ into two classes without non-zero homomorphisms between objects in the different classes. For an introduction to Auslander-Reiten theory in the setting of triangulated categories see [4]. Note that, according to [8], the existence of Auslander-Reiten triangles in $\mathcal{C}$ is equivalent to the existence of a Serre functor on $\mathcal{C}$.

2. Additive functions

Let Γ be a stable component of the Auslander-Reiten quiver of $\mathcal{C}$. We will say that a function $\phi : \Gamma_0 \to \mathbb{Z}$ is *additive* if on each Auslander-Reiten triangle $U \to (V_1 \oplus \cdots \oplus V_r) \to W \to U[1]$ whose first three terms lie in Γ_0 we have $\phi(U) + \phi(W) = \phi(V_1) + \cdots + \phi(V_r)$. We will refer to these first three terms as a *mesh* of the quiver. We say that ϕ is *positive* if it takes non-negative values, and somewhere is positive. We say that ϕ is *defective* on this mesh *of defect* $d > 0$ if $\phi(U) + \phi(W) = \phi(V_1) + \cdots + \phi(V_r) + d$. We regard Γ as a *valued Riedtmann quiver* in the terminology of [5], so that $\Gamma \cong \mathbb{Z}T/G$ is a quotient of $\mathbb{Z}T$ where T is a valued tree, ϕ lifts to a function on the vertices of $\mathbb{Z}T$, which is additive or defective according as ϕ is additive or defective on Γ.

When T is a finite valued tree it defines a Coxeter group W, acting on a vector space $\mathbb{Q}^{T_0}$ with basis indexed by the vertices of T. This is explained at the start of [2] and we will follow the arguments presented there. A *slice* of $\mathbb{Z}T$ is a connected subgraph whose vertices are a set of representatives for the τ-orbits in Γ. Such an S has the same underlying valued graph as T. When S is a slice of $\mathbb{Z}T$ and $f : (\mathbb{Z}T)_0 \to \mathbb{Z}$ is a function, the values $\{f(x) \bigm| x \in S_0\}$ may be regarded as the coordinates of a vector in $\mathbb{Q}^{T_0}$, which we denote $f(S)$.

The next result is well known. Part (3) is a strengthening of a result which appears as [9, Cor. 2.4]. We supply some proofs for the convenience of the reader.

Lemma 2.1. *Let T be a finite valued tree, S a slice of $\mathbb{Z}T$ and c a Coxeter transformation in the corresponding Coxeter group W.*

1. *If f is an additive function on $\mathbb{Z}T$ then $f(\tau S) = cf(S)$.*
2. *If T is a Dynkin tree there is no positive additive function on $\mathbb{Z}T$.*
3. *If T is an extended Dynkin diagram then positive additive functions on $\mathbb{Z}T$ are periodic, of bounded period (depending on the diagram).*

Proof. (1) This is well-known; a possible reference is [9, Lemma 2.3].

Parts (2) and (3) follow from the discussion in [2] leading to the proof of Lemma 1.7. In both cases we let f be a positive additive function and observe that $f(S)$ is a positive vector in $\mathbb{Q}^{T_0}$, all of whose images under the powers of c are positive, by (1). If T is a Dynkin diagram then W, and hence c, have finite order so $f(S)$ lies in a finite orbit of vectors under the action of the powers of c. The sum of these vectors is again a positive vector, and it is fixed by c. By [2, Lemma 1.3] there are no such vectors. Hence no such f exists.

For (3) we suppose that T is an extended Dynkin diagram and write N for the fixed point space of W on $\mathbb{Q}^{T_0}$. It follows from Prop. 1.2 and Prop. 1.5 of [2] in this case (and also for Dynkin diagrams) that the action of W on $\mathbb{Q}^{T_0}/N$ is as a finite group $\overline{W}$. Writing m for the order of the image of c in $\overline{W}$ it is immediate that $c^m f(S) = f(S) + \mathbf{v}$ for some vector $\mathbf{v}$ in N so that $(c^m)^t f(S) = f(S) + t\mathbf{v}$ for all $t \in \mathbb{Z}$. If $\mathbf{v} \neq 0$ it is possible to choose t so that some coordinate of $(c^m)^t f(S)$ is negative, which is a contradiction. We deduce that $\mathbf{v} = 0$ and hence that $c^m f(S) = f(S)$. This shows that f is periodic, of period dividing m. □

For each object M of $\mathcal{C}$ we let f_M and g_M be the functions defined on objects N of $\mathcal{C}$ by $f_M(N) = \dim \operatorname{Hom}_{\mathcal{C}}(M,N)$ and $g_M(N) = \dim \operatorname{Hom}_{\mathcal{C}}(N,M)$. The fact that these functions are additive on almost all of $\mathcal{C}$ underlies our proofs of the main theorems. In the context of stable module categories of self-injective algebras this fact has been observed previously by Erdmann and Skowroński [3, Lemma 3.2].

Lemma 2.2. *Let Γ be a stable Auslander-Reiten quiver component and let M be an indecomposable object. Let $d := \dim \operatorname{End}(M)/\operatorname{Rad}\operatorname{End}(M)$.*

1. *f_M is additive on every mesh of Γ except the meshes with M and $M[1]$ as the right hand object.*
2. *g_M is additive on every mesh of Γ except the meshes with M and $M[-1]$ as the left hand object.*

On the excluded meshes the functions are defective, of defect d if $M \ncong M[1]$ and of defect $2d$ if $M \cong M[1]$.

Proof. This follows from [10, Proposition 3.1]. We present the argument for f_M here. The argument for g_M is dual.

Let $U \to V \to W \to U[1]$ be an Auslander-Reiten triangle. In the long exact sequence

$$\cdots \xrightarrow{\beta_{-1}} \operatorname{Hom}(M, W[-1])$$
$$\xrightarrow{\delta_{-1}} \operatorname{Hom}(M,U) \xrightarrow{\alpha_0} \operatorname{Hom}(M,V) \xrightarrow{\beta_0} \operatorname{Hom}(M,W)$$
$$\xrightarrow{\delta_0} \operatorname{Hom}(M,U[1]) \xrightarrow{\alpha_1} \cdots$$

the Auslander-Reiten lifting property of the triangle shows that β_0 is epi and hence $\delta_0 = 0$ except when $M \cong W$, and β_{-1} is epi and hence $\delta_{-1} = 0$ except when $M \cong W[-1]$. Thus, unless $W \cong M$ or $M[1]$, the middle sequence is a short exact sequence, showing that f_M is additive on this mesh in this case. When $W \cong M$ but $M \ncong M[1]$ then $M \ncong W[-1]$ and so the only δ which is non-zero is δ_0, which has rank d. This shows that f_M has defect d on this mesh. The argument is similar when $M \cong W[-1]$. When $M \cong M[1] \cong W$ both δ_{-1} and δ_0 have rank d, so that f_M has defect $2d$ on this mesh. □

3. Proof of Theorem 1.1

The proof of Theorem 1.1 proceeds in three steps. We suppose that Γ is a stable component with tree class T, so that $\Gamma \cong \mathbb{Z}T/G$ for some group G.

Step 1: We show that every object in $\mathcal{C}$ is the shift of an object in Γ:

$$\mathcal{C} = \bigcup_{i\in\mathbb{Z}} \Gamma[i].$$

For, if not, there is an indecomposable object $M \notin \bigcup_{i\in\mathbb{Z}} \Gamma[i]$ with either $f_M \neq 0$ or $g_M \neq 0$ on Γ. Let us say $f_M \neq 0$, as the argument is similar if $g_M \neq 0$. Then f_M is positive and additive on Γ and hence gives a positive additive function on $\mathbb{Z}T$, which is not possible if T is Dynkin by Lemma 2.1(2). This impossibility establishes the assertion.

Step 2: We show that when Γ is finite it is closed under shift, and hence $\mathcal{C}$ is locally finite. To prove this, suppose that Γ is finite. If there are only finitely many distinct shifts $\Gamma[i]$ then, by Step 1, $\mathcal{C}$ has only finitely many indecomposable objects and so is locally finite. From this it follows that $\mathcal{C}$ has only one component, by [11, Theorem 2.3.5]. Otherwise, if there are infinitely many distinct $\Gamma[i]$, it follows that all the $\Gamma[i]$ must be distinct. Let M be an indecomposable object, so that $M \in \Gamma[i]$ for some i. By Lemma 2.2 f_M only fails to be additive on $\Gamma[i]$ and $\Gamma[i+1]$, so by Lemma 2.1 it is only non-zero on these components, and similarly g_M is only non-zero on $\Gamma[i]$ and $\Gamma[i-1]$. This shows that $\mathcal{C}$ is locally finite, and so there is only one quiver component, again by [11, Theorem 2.3.5].

Step 3: We show that when Γ is infinite it is closed under shift and locally finite. Assuming Γ is infinite we must have $\Gamma \cong \mathbb{Z}T$, because factoring out any non-identity admissible group of automorphisms of $\mathbb{Z}T$ gives a finite quiver (see, for instance, [12, Sec. 3]). Let M be any object in Γ. Then f_M is non-zero on Γ (because $f_M(M) \neq 0$), and f_M is additive everywhere on Γ except on the mesh whose right hand term is M, and also on the mesh whose right hand term is $M[1]$ (if it happens to lie in Γ). To the left and right of these meshes f_M is periodic, because the Coxeter transformation has finite order. From each of these periodic regions we may extend f_M to a periodic, non-negative additive function on the whole of Γ, which must be zero. Thus f_M is zero both to the left and to the right of M and $M[1]$. At this point we may deduce that $M[1]$ does lie in Γ because otherwise we would deduce that f_M must be 0 at M, and also by the same reasoning that $M \neq M[1]$. We conclude that $\Gamma[1] = \Gamma$, so Γ is closed under shift.

Furthermore we have seen that f_M is non-zero only on objects which lie between M and $M[1]$, which is a finite set of indecomposable objects. Similarly g_M is non-zero only on a finite set of indecomposable objects. This shows that $\mathcal{C}$ is locally finite and completes the proof of Theorem 1.1.

It is not part of the proof, but we may note that the dimensions of Hom spaces between M and other objects in $\mathcal{C}$ are now completely determined as the unique function f_M which is additive everywhere except on the meshes terminating at M and $M[1]$, where it has defect d.

4. Proof of Theorem 1.3

We suppose that Γ_1 and Γ_2 are stable components of the Auslander-Reiten quiver of $\mathcal{C}$ and that Γ_1 has an extended Dynkin diagram as its tree class.

If Γ_2 is a shift of Γ_1 then it has the same tree class, which is an extended Dynkin diagram, and we are done. Thus we may suppose that Γ_2 is not a shift of Γ_1 and it follows, by Lemma 2.2 that for each M in Γ_2 the functions f_M and g_M are additive on Γ_1. They are also non-negative, and hence by Lemma 2.1 they are always periodic, of bounded period.

Since $\mathrm{Hom}(\tau^n M, X) \cong \mathrm{Hom}(M, \tau^{-n} X)$ it now follows that all functions f_X and g_X are periodic on Γ_2 when $X \in \Gamma_1$, of bounded period. We can find a non-zero such function. This enables us to put a non-negative additive function on the tree of Γ_2, which implies that it must be one of the trees listed, by [5].

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