Two Classifications of Simple Mackey Functors with Applications to Group Cohomology and the Decomposition of Classifying Spaces

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With best wishes to my teacher, Karl Gruenberg, on his 65th birthday

In previous joint work with Thévenaz we have classified simple Mackey functors [15] and developed a structural theory of Mackey functors which takes its standpoint from representation theory [16]. In this paper we show that this type of theory may be used to perform specific calculations of Mackey functor values, and also to obtain structural results about the decomposition of classifying spaces. It turns out in doing this that it is profitable to allow the precise nature of the Mackey functors we consider to vary, so as best to suit the problem in hand.

Our first application is to group cohomology $H^n(G,k)$ with a trivial coefficient module. This gives a Mackey functor $H^n(\ ,k)$ defined on every finite group G, and provides an example of what we will call a 'global Mackey functor', namely a Mackey functor which has a consistent definition on all finite groups. Our approach is to consider a composition series as a global Mackey functor of $H^n(\ ,k)$, by which we mean a series of global Mackey functors

$$\cdots \subset M_{i-1} \subset M_i \subset M_{i+1} \subset \cdots \subset H^n(\ ,k)$$

such that $\bigcap M_i = 0$, $\bigcup M_i = H^n(\ ,k)$ and M_{i+1}/M_i is always a simple global Mackey functor. At this point the question of the existence of such series arises, and we show that group cohomology does indeed have such a series. We show also that the multiplicities of the composition factors are determined independently of the choice of series. This sequence of multiplicities of simple global Mackey functors as composition factors is now an invariant of group cohomology which it is of interest to compute.

For our application to the computation of group cohomology we rely on two more things. The first is the fact that the composition factor multiplicities can be computed from a knowledge only of p-groups. The second is a classification of the simple global Mackey functors, which is the first result we present in this paper. The arguments we use parallel those in [15]. By virtue of an explicit formula for these simple objects, knowing their multiplicities as composition factors we are able to give the dimensions of cohomology groups. A useful feature of this approach is that since the computation is done using a knowledge only of p-groups, we obtain simultaneously a formula for the mod p cohomology of all groups with a given Sylow p-subgroup, in terms of the cohomology of the Sylow p-subgroup and the fusion of certain subgroups.

Turning to our second application of this type of theory, we show how a study of Mackey functors yields a proof of the theorem of Benson-Feshbach [1] and Martino-Priddy [8] on the stable decomposition of classifying spaces. For this we need to consider global Mackey functors with extra structure analogous to the that of the inflation map in group cohomology. To avoid a potentially longer name for these objects we will call them simply 'inflation functors'. They were used by Symonds [13] where they are called 'functors with Mackey structure', and they are defined in [4, p.278] where they are called 'global Mackey functors' (thus differing from our use of the term). The term 'Burnside functor' is also used by some to refer to a dual notion. Inflation functors are Mackey functors defined on all groups with the extra specification that for each group homomorphism $\theta : H \to G$ (not just monomorphism) there is a contravariant map
$M^*(\theta) : M(G) \to M(H)$. The classification of the simple inflation functors proceeds in a similar way to that for simple global Mackey functors, and it so happens that these two types of simple object are in bijection, although their detailed structure is different.

The crucial key in applying this theory to the stable decomposition of classifying spaces is Carlsson's verification of the Segal conjecture, which by work of Lewis, May and McClure [6] implies for a finite group G that the stable wedge decompositions of the suspension spectrum BG_+ biject with decompositions of the identity in $A(G,G)^\wedge$ as a sum of orthogonal idempotents (see also [1, 8, 10]). Here $A(G,G)$ is the double Burnside ring and $A(G,G)^\wedge$ is its completion at the augmentation ideal of the Burnside ring. We will see that when $A(\ \ ,G)$ is regarded as a functor in the first variable, it is projective in the category of inflation functors. In fact it is a representable functor, and its endomorphism ring is $A(G,G)$. We will fix a prime and work over the p-adic integers $\mathbb{Z}_p$, in which case when G is a p-group the stable summands of the p-completed spectrum $(BG_+)^\wedge_p$ biject with the summands of the functor $A(\ \ ,G) \otimes \mathbb{Z}_p$. We will show (working over $\mathbb{Z}_p$) that the indecomposable summands of these representable functors are the projective covers of the simple inflation functors, and so their isomorphism types are in bijection with the isomorphism types of the simple functors. Their multiplicities as summands of $A(\ \ ,G)\otimes\mathbb{Z}_p$ are immediately computable from the simple functors, and for arbitrary finite groups G these numbers are also the multiplicities as summands of the indecomposable stable p-completed summands of $(BG_+)^\wedge_p$. In this way using an explicit formula for the simple inflation functors we recover the computational approach to these multiplicities given in [1]. What our approach also gives is a structural interpretation of these numbers, since a single simple inflation functor contains the multiplicity information for the corresponding classifying space summand, as a summand of $(BG_+)^\wedge_p$ for every group G. We obtain also an equivalence of categories in which stable p-completed classifying space summands correspond to indecomposable projective inflation functors. The philosophy behind this approach is that the summands of $(BG_+)^\wedge_p$ are replaced by projective objects in a category which have very nice properties making them easy to deal with.

Thanks are due to many people for their technical assistance in connection with this work. I thank Dave Benson and Mark Feshbach; and especially Nick Kuhn, for an extremely valuable conversation concerning the application of Mackey functors to the decomposition of classifying spaces. At a time when the connection was still mysterious to me he was able to sketch in outline the way the theory would probably go, and with some technical modifications that is the approach presented here.

1. Global Mackey functors and inflation functors

In this section we will make the definitions of two very similar kinds of Mackey functor. The differences are apparently slight and much of the theory for one kind works very much the same way as for the other. These differences will become more apparent when we obtain explicit formulae for the functors, and it will turn out that one type of Mackey functor is harder to compute with, but is nevertheless necessary for the application we have in mind.

A *global Mackey functor* over a commutative ring R is given by the following specification. For each finite group G there is an R-module $M(G)$ and for every monomorphism of groups $\alpha : H \to G$ there are R-module homomorphisms

$$M_*(\alpha) : M(H) \to M(G)$$
$$M^*(\alpha) : M(G) \to M(H)$$

such that

(1) $M_*(\alpha\beta) = M_*(\alpha)M_*(\beta)$ and $M^*(\alpha\beta) = M^*(\beta)M^*(\alpha)$ always,
(2) whenever $\alpha : H \to H$ is an inner automorphism then $M_*(\alpha) = M^*(\alpha) = 1$,
(3) whenever $\alpha : H \to H$ is an automorphism then $M_*(\alpha^{-1}) = M^*(\alpha)$, and
(4) for every pair of monomorphisms $H \xrightarrow{\alpha} G \xleftarrow{\beta} K$ we have

$$M^*(\beta)M_*(\alpha) = \sum_{x \in [\alpha(H)\backslash G/\beta(K)]} M_*(\phi_x)M^*(\psi_x)$$

where

$$\phi_x : \alpha(H) \cap {}^x\beta(K) \xrightarrow{c_x} \alpha(H)^x \cap \beta(K) \hookrightarrow \beta(K) \xrightarrow{\beta^{-1}} K$$

and

$$\psi_x : \alpha(H) \cap {}^x\beta(K) \hookrightarrow \alpha(H) \xrightarrow{\alpha^{-1}} H.$$

In condition (4) we are using the notation that $[X]$ denotes a set of representatives for the elements in a quotient object X: thus $\alpha(H)\backslash G/\beta(K)$ is a set of double cosets and $[\alpha(H)\backslash G/\beta(K)]$ a set of representatives for these double cosets, consisting of elements of G.

A morphism $\theta : M_1 \to M_2$ of global Mackey functors is a collection of R-module homomorphisms $\theta : M_1(G) \to M_2(G)$ for every finite group G which is natural with respect to both M_* and M^*. Thus whenever $\alpha : H \to G$ is a group monomorphism, the squares

$$\begin{array}{ccc} M_1(H) & \stackrel{M_*(\alpha)}{\longrightarrow} & M_1(G) \\ \big\downarrow \theta & & \big\downarrow \theta \\ M_2(H) & \stackrel{M_*(\alpha)}{\longrightarrow} & M_2(G) \end{array}$$

and

$$\begin{array}{ccc} M_1(G) & \stackrel{M^*(\alpha)}{\longrightarrow} & M_1(H) \\ \big\downarrow \theta & & \big\downarrow \theta \\ M_2(G) & \stackrel{M^*(\alpha)}{\longrightarrow} & M_2(H) \end{array}$$

must commute. Global Mackey functors over R form an abelian category, which will be denoted Mack_R. In cases where it is clear what R is without further specification, we may simply write Mack. We will speak of subfunctors, quotient functors etc., and in particular we have the notion of simple global Mackey functors, namely those which have no non-trivial subfunctors.

We now define the notion of an inflation functor. We say that M is an *inflation functor over* R provided that for each group G there is an R-module $M(G)$ together with R-module homomorphisms

$$M_*(\alpha) : M(H) \to M(G)$$
$$M^*(\gamma) : M(G) \to M(H)$$

whenever $\alpha : H \to G$ is a monomorphism of groups and $\gamma : H \to G$ is an arbitrary group homomoprhism. These must satisfy conditions (2), (3) and (4) previously given for global Mackey functors, together with

(1′) $M_*(\alpha\beta) = M_*(\alpha)M_*(\beta)$ for all monomorphisms α, β, and
$M^*(\gamma\delta) = M^*(\delta)M^*(\gamma)$ for all homomorphisms γ, δ.

(5) Whenever

$$\begin{array}{ccc} G' & \stackrel{\gamma}{\longrightarrow} & G \\ {\scriptstyle i'}\big\uparrow & & \big\uparrow{\scriptstyle i} \\ H' & \stackrel{\delta}{\longrightarrow} & H \end{array}$$

is a commutative diagram of groups in which γ and δ are surjective and i, i' are inclusions with $H' = \gamma^{-1}(H)$, then $M^*(\gamma)M_*(i) = M_*(i')M^*(\delta)$.

Morphisms of inflation functors are families of R-module homomorphisms natural with respect to both M_* and M^*. Inflation functors form an abelian category which we will denote Mack^*_R. In cases where it is clear what R is without further specification we will simply write Mack^*.

We should point out at this stage that there is a dual notion in which the covariant part of a global Mackey functor is extended to all group homomorphisms but the contravariant part is not. We will not explicitly consider these covariant versions of inflation functors in this paper, but they also form an abelian category which one might denote Mack_*. So, for example, in any fixed dimension n the cohomology $H^n(G,R)$ is a functor in Mack^*, while homology $H_n(G,R)$ is a functor in Mack_*. In general, the dual of an object in Mack^* is an object in Mack_* (using 'dual' in the manner of [16]) and vice versa, so that properties of one category may be deduced from those of the other. By this means the classification of the simple objects in Mack^* which we are about to give also yields a classification of the simple objects in Mack_*.

It is clear that every inflation functor yields a global Mackey functor by restricting the definition of the contravariant part M^* to group monomorphisms, instead of arbitrary homomorphisms. It is also evident that if we restrict a global Mackey functor to a particular group G and its subgroups, together with the monomorphisms formed by inclusion of subgroups and conjugations by elements of G, we obtain a Mackey functor on G. Because of this connection with Mackey functors, if $\alpha : H \to G$ happens to be an inclusion of subgroups, we may write I_H^G for $M_*(\alpha)$ and R_H^G for $M^*(\alpha)$, and if $\alpha : H \to {}^gH$ is conjugation by $g \in G$ we write c_g for $M_*(\alpha) = M^*(\alpha^{-1})$. Since every monomorphism of groups may be expressed as an isomorphism followed by an inclusion of subgroups, it would be equivalent (in the presence of the other axioms) to replace axiom (4) by
(4′) for every pair of subgroups H, K of G,

$$R_K^G I_H^G = \sum_{g \in [K \backslash G/H]} I_{K \cap {}^gH}^K c_g R_{K^g \cap H}^H$$

which is just the usual Mackey decomposition formula. Another useful simplification of notation is that we may write α_* instead of $M_*(\alpha)$ and α^* instead of $M^*(\alpha)$.

Notice that for any group H the evaluation $M(H)$ of a global Mackey functor or inflation functor has the structure of an $R[\mathrm{Out}(H)]$-module. For the mapping

$$M_* : \mathrm{Aut}(H) \to \mathrm{Aut}(M(H))$$

specified by $\alpha \mapsto M_*(\alpha)$ is a group homomorphism, and the axiom that $M_*(\alpha) = 1$ when α is inner, means that in fact we have an action of $\mathrm{Out}(H)$ on $M(H)$.

In some situations it is a technical advantage to have an alternative but equivalent definition of global Mackey functors and inflation functors, as R-additive functors defined on certain categories Ω_R and Ω_R^+ which we now define. The ingredients of this approach may be found in the work of Lewis, May and McClure [6, 10] and tom Dieck [4, p.278]. As a first step we define categories ω and ω^+, in both of which the objects are taken to be all finite groups. A morphism $G \to H$ in ω is a $G \times H$-set X, taken up to $G \times H$-set isomorphism, which is free on restriction to each of G and H. It is convenient to regard G as acting from the left and H as acting from the right, and we may write ${}_GX_H$

to indicate this situation. Composition of morphisms is now an amalgamated product ${}_GX_H \underset{H}{\times} {}_HY_K = X \times Y/\sim$ where $(xh, y) \sim (x, hy)$ for all $x \in X$, $y \in Y$ and $h \in H$. In ω^+ we take the morphisms $G \to H$ to be isomorphism classes of $G \times H$ sets ${}_GX_H$ which are free on restriction to H. Composition is defined in the same way as for ω. In both ω and ω^+ the morphisms have the structure of an abelian monoid given by the binary operation of disjoint union: ${}_GX_H +_G Y_H =_G X_H \cup_G Y_H$. In fact the morphisms $G \to H$ form a *free* abelian monoid with basis the transitive $G \times H$-sets with the appropriate free property. This is because each $G \times H$-set is uniquely expressible as the disjoint union of its orbits. We may thus embed $\mathrm{Hom}_\omega(G, H)$ and $\mathrm{Hom}_{\omega^+}(G, H)$ into free abelian groups $\mathbb{Z}\,\mathrm{Hom}_\omega(G, H)$ and $\mathbb{Z}\,\mathrm{Hom}_{\omega^+}(G, H)$ by the usual universal construction. We now define the categories Ω_R and Ω_R^+. These again have the finite groups as their objects, and we take

$$\mathrm{Hom}_{\Omega_R}(G, H) = \mathbb{Z}\,\mathrm{Hom}_\omega(G, H) \otimes R$$
$$\mathrm{Hom}_{\Omega_R^+}(G, H) = \mathbb{Z}\,\mathrm{Hom}_{\omega^+}(G, H) \otimes R.$$

The group $\mathrm{Hom}_{\Omega_\mathbb{Z}^+}(G, H)$ appears in connection with the Segal conjecture ([6], [1], [8]) where it is denoted $A(G, H)$. The special case $A(G, G)$ is called the *double Burnside ring.* We will find it convenient to introduce the notation $A_R(G, H)$ for $\mathrm{Hom}_{\Omega_R^+}(G, H)$, so that $A_\mathbb{Z}(G, H) = A(G, H)$.

We may now give alternative definitions of our Mackey functors as follows: a global Mackey functor is an R-additive functor $M : \Omega_R \to R$-mod. An inflation functor is an R-additive functor $M : \Omega_R^+ \to R$-mod. We remark immediately that because Ω_R is isomorphic to its opposite category it does not matter if we take M to be covariant or contravariant in defining a global Mackey functor. On the other hand this does make a difference with Ω_R^+: a covariant R-additive functor $\Omega_R^+ \to R$-mod gives us an object of Mack_*, while the contravariant functors give objects of Mack^*.

To explain the connection between the different definitions we show first how each R-additive functor $\Omega_R^+ \to R$-mod gives rise to an inflation functor in the sense previously defined. Suppose $F : \Omega_R^+ \to R$-mod is a contravariant R-additive functor. We define an inflation functor by putting $M(G) = F(G)$ for all groups G. If $\alpha : H \to G$ is a monomorphism and $\gamma : H \to G$ is any group homomorphism we factor α as $H \xrightarrow{\alpha'} H' \overset{\iota}{\hookrightarrow} G$ where α' is an isomorphism and put $M_*(\alpha) = F({}_GG_H)$ where H acts on G via $(\alpha')^{-1}$ and $M^*(\gamma) = F({}_HG_G)$ where H acts on G via γ.

To go in the opposite direction we show that a functor defined on Ω_R^+ may be recovered from an inflation functor, we first make some comments about the morphisms in ω and ω^+. Any morphism in ω or ω^+ is a sum of transitive $G \times H$-sets, each of which is isomorphic to a set of the form $(G \times H)/\Delta$ where $\Delta \le G \times H$. As explained in [1], the condition that this set is free on restriction to H is equivalent to the requirement that $\Delta = \{(g, \phi(g)) \mid g \in K\}$ where K is some subgroup of G and ϕ is some homomorphism $K \to H$. The condition that the set is also free on restriction to G is now equivalent to the requirement that ϕ is a monomorphism. One sees that $(G \times H)/\Delta \cong {}_GG_K \underset{K}{\times} {}_KH_H$ where K acts on the left of H via ϕ. Now given an inflation functor M in the first sense we define $F(G) = M(G)$ for all groups G, and put $F(G \times H/\Delta) = M_*(\iota)M^*(\phi)$ where $\iota : K \hookrightarrow G$ is inclusion. Extending this definition by R-linearity we obtain a contravariant functor $\Omega_R^+ \to R$-mod. The translation procedure we have just described works for inflation functors and Ω_R^+, but if we simply require that all our group homomorphisms are monomorphisms we obtain the translation between global Mackey functors as first defined and R-additive functors defined on Ω_R. To verify that these translations are valid, we check that the relations satisfied by the Mackey functor morphisms are also satisfied in ω and ω^+, and that all relations on morphisms in ω and ω^+ are deducible from these.

The construction of the categories Ω_R and Ω_R^+ gives rise to a way of looking at the functors defined on them as modules over certain algebras. Let us first replace Ω_R and Ω_R^+ by skeletal subcategories $\overline{\Omega_R}$ and $\overline{\Omega_R^+}$ in which the objects are representatives of the isomorphism types of groups. We put

$$\mu_R = \bigoplus_{G,H} \operatorname{Hom}_{\overline{\Omega_R}}(G,H)$$
$$\mu_R^+ = \bigoplus_{G,H} \operatorname{Hom}_{\overline{\Omega_R^+}}(G,H).$$

By defining the product of two morphisms to be their composite if they can be composed, and zero otherwise, these two R-modules acquire the structure of R-algebras. For any ring Λ we define Λ-mod to be the category of Λ-modules. The following is immediate, as in [16].

(1.1) PROPOSITION. Mack_R *is equivalent to* μ_R*-mod, and* Mack_R^* *is equivalent to* $\mu_R^{+\mathrm{op}}$*-mod.*

We take the opposite ring $\mu_R^{+\mathrm{op}}$ here because the objects in Mack_R^* are contravariant functors on Ω_R^+. We call μ_R the *global Mackey algebra* over R. Recall from [16] that in the context of Mackey functors defined on a fixed group G we also defined an algebra $\mu_R(G)$ called the *Mackey algebra* which plays a rôle analogous to that of μ_R and μ_R^+ with respect to the category $\mathrm{Mack}_R(G)$ of Mackey functors defined on G. For each finite group G there is an R-algebra homomorphism $\mu_R(G) \to \mu_R$ (which will never be unital, because μ_R has no identity) specified as follows. Using the notation of [16], $\mu_R(G)$ has as a basis the products $I_{^gL}^K c_g R_L^H$ where $H, K \leq G$, g is a double coset representative from $K\backslash G/H$ and $L \leq H \cap K^g$ is taken up to $H \cap K^g$-conjugacy. We define

$$\mu_R(G) \to \mu_R$$
$$I_{^gL}^K c_g R_L^H \mapsto (H \times K)/\Delta$$

where $\Delta = \{(h, {}^gh) \bigm| h \in L\}$. Now the restriction functor $\mathrm{Mack}_R \to \mathrm{Mack}_R(G)$ corresponds, when we regard functors as modules, to restricting the action of μ_R along the homomorphism $\mu_R(G) \to \mu_R$ and multiplying by $1 \in \mu_R(G)$.

(1.2) PROPOSITION. *The restriction functor* $\mathrm{Mack}_R \to \mathrm{Mack}_R(G)$ *has a left adjoint.*

Proof. In module notation the left adjoint is $M \mapsto \mu_R \otimes_{\mu_R(G)} M$, where M is a Mackey functor defined on G. □

We mention also that by a similar argument the restriction functor $\mathrm{Mack}_R^* \to \mathrm{Mack}_R$ has a left adjoint. While this approach gives an easy proof of the theoretical existence of the left adjoint, it is not so easy to describe it in explicit terms. A major part of our work will be concerned with obtaining explicit formulae for the values of related adjoints on a certain special type of Mackey functor.

2. The classification of simple functors

We will show that both the simple global Mackey functors and also the simple inflation functors are in bijection with the set of pairs (H, V) where H is a group and V is a simple $R[\mathrm{Out}(H)]$-module, both taken up to isomorphism. Many of the arguments for these two cases run in parallel, but we will see when we produce explicit formulae that the detailed structure of these two types of functor is different. To describe the simple functors we will use the terminology that a *minimal group* for either a global Mackey functor or an inflation functor M is a group H such that $M(H) \neq 0$ but $M(K) = 0$ for every proper subgroup K of H.

(2.1) PROPOSITION. *Let* S *be either a simple global Mackey functor or a simple inflation functor. Then* S *has a unique isomorphism class of minimal groups* H*. Furthermore,* $S(H)$ *is a simple* $R[\mathrm{Out}(H)]$*-module.*

Proof. We give the proof explicitly in the case of inflation functors, and at the same time prove the result for global Mackey functors by indicating in italics the words in the proof that must be changed. In every case the word in italics is a homomorphism which must be supposed in addition to be a monomorphism to obtain the proof for global Mackey functors. Thus the word *homomorphism* must be changed to monomorphism, and *epimorphism* must be changed to isomorphism.

Let H be any minimal group of S, chosen to have minimal order. Let $W \subseteq S(H)$ be an $R[\mathrm{Out}(H)]$-submodule of $S(H)$. For each group K consider

$$T(K) = \{x \in S(K) \mid S^*(\alpha)(x) \in W \text{ for all } \textit{homomorphisms } \alpha : H \to K\}.$$

We claim that T is a subfunctor of S. first, if $H \xrightarrow{\alpha} J \xrightarrow{\theta} K$ are any *homomorphisms* and $x \in T(K)$ then $S^*(\alpha)S^*(\theta)(x) = S^*(\theta\alpha)(x) \in W$ shows that $S^*(\theta) : T(K) \to T(J)$. If θ happens to be an isomorphism then this shows also that $S_*(\theta) : T(J) \to T(K)$ since

$S_*(\theta) = S^*(\theta^{-1})$ in this case, and we can apply the previous argument to θ^{-1}. Thus it remains to show that if $i : J \hookrightarrow K$ is an inclusion of subgroups then

$$I_J^K = S_*(i) : T(J) \to T(K).$$

Let $y \in T(J)$ and suppose $H \xrightarrow{\beta} \beta(H) \xrightarrow{j} K$ is some *homomorphism* expressed as a composite of an *epimorphism* β and an inclusion j. Then

$$\begin{aligned} S^*(j\beta)I_J^K(y) &= S^*(\beta)R^K_{\beta(H)}I_J^K(y) \\ &= S^*(\beta) \sum_{g \in [\beta(H)\backslash K/J]} I^{\beta(H)}_{\beta(H)\cap {}^gJ}\, c_g\, R^J_{\beta(H)^g \cap J}(y) \\ &= S^*(\beta) \sum_{\substack{g \in [\beta(H)\backslash K/J] \\ \beta(H)^g \subseteq J}} c_g\, R^J_{\beta(H)^g}(y) \\ &\in W \end{aligned}$$

since $S(\beta(H)^g \cap J) = 0$ unless $\beta(H)^g \subseteq J$, by minimality of H, and the remaining terms are $S^*(\gamma)(y)$ for some *homomorphism* γ.

We have $T(H) = W$, so if $W \neq 0$ then $T = S$ by simplicity of S, and so $S(H) = W$ in this case. Thus $S(H)$ must be simple.

If K is a group with no subgroup isomorphic to H then every homomorphism $\alpha : H \to K$ must factor through a group of smaller size than H, on which S vanishes, and so $S^*(\alpha)S(K) = 0$. Thus $T(K) = S(K)$ no matter what the choice of W. In particular this holds when $W = 0$, in which case T is a proper subfunctor of S, so T is the zero functor. Thus $S(K) = T(K) = 0$, and we deduce that the unique minimal group of S is H (up to isomorphism). □

The next stage in the classification of the simple global Mackey functors and inflation functors is to show for each pair (H, V) consisting of a group H and a simple $R[\mathrm{Out}(H)]$-module V that there is a simple global Mackey functor $\mathcal{S}$ with minimal group H and $\mathcal{S}(H) \cong V$, and that there is a simple inflation functor $\mathbb{S}$ with minimal group H and $\mathbb{S}(H) \cong V$. The construction depends on the properties of a pair of adjoint functors which we now describe.

For each group H there are functors

$$\mathcal{F} : \mathrm{Mack} \to R[\mathrm{Out}(H)]\text{-mod}$$

and

$$\mathcal{F} : \mathrm{Mack}^* \to R[\mathrm{Out}(H)]\text{-mod}$$

both given by

$$M \mapsto \overline{M}(H)$$

where $\overline{M}(H) = M(H) \,/\, \sum_{K<H} I_K^H M(K)$. We now construct functors

$$\mathcal{G} : R[\mathrm{Out}(H)]\text{-mod} \to \mathrm{Mack}$$

and

$$\mathcal{G} : R[\mathrm{Out}(H)]\text{-mod} \to \mathrm{Mack}^*$$

which are the right adjoints of the functors $\mathcal{F}$. We use the same symbols $\mathcal{F}$ and $\mathcal{G}$ in the two cases of global Mackey functors and inflation functors because in each case the formulae which define the functors are the same. We will in fact define an inflation functor $\mathcal{G}(V) = J_{H,V} \in \mathrm{Mack}^*$ whose restriction as a global Mackey functor is $\mathcal{G}(V) \in \mathrm{Mack}$. I am grateful to Peter Symonds for pointing out the approach presented here, which is more straightforward and elegant than my original construction.

We first define $J_{H,V}$ on groups L which are isomorphic to H, by putting $J_{H,V}(L) = V$ as a set. We now define $J_{H,V}$ on isomorphisms of these groups. For each group $L \cong H$ let us fix an isomorphism $\alpha : H \to L$. Suppose that we are given two such groups and fixed isomorphisms $\alpha_i : H \to L_i$, $i = 1, 2$, and also some isomorphism $\gamma : L_1 \to L_2$. We define

$$(J_{H,V})_*(\gamma)(v) = (\alpha_2^{-1}\gamma\alpha_1) \cdot v$$

and $(J_{H,V})^*(\gamma) = (J_{H,V})_*(\gamma)^{-1}$. In particular this specification makes $J_{H,V}(L)$ into an $R[\mathrm{Aut}(L)]$-module, and hence an $R[\mathrm{Out}(L)]$-module, which is simply V with the action transported by $\alpha : H \to L$. To shorten the notation we will denote this module simply by ${}^{\alpha}V$.

We now make the general definition of $J_{H,V}$. For each group G we form the direct sum

$$\bigoplus_{\substack{L \le G \\ \alpha : H \cong L}} {}^{\alpha}V$$

taken over all subgroups of G isomorphic to H, where for each such subgroup we have chosen an isomorphism α. There is an action of G on this direct sum given as follows. For elements $v \in {}^{\alpha}V$ and $g \in G$ conjugation gives an isomorphism $c_g : L \to {}^{g}L$ and the effect of g on v is $(J_{H,V})_*(c_g)(v)$, which lies in the summand corresponding to ${}^{g}L$. To define $J_{H,V}$ on G we take the fixed points

$$J_{H,V}(G) = \Big(\bigoplus_{\substack{L \le G \\ \alpha : H \cong L}} {}^{\alpha}V \Big)^{G}.$$

If $\iota : K \hookrightarrow G$ is an inclusion of a subgroup we define $J_{H,V*}(\iota) = I_K^G$ to be the composite map

$$I_K^G : \Big(\bigoplus_{\substack{L \le K \\ \alpha : H \cong L}} {}^{\alpha}V \Big)^{K} \hookrightarrow \Big(\bigoplus_{\substack{L \le G \\ \alpha : H \cong L}} {}^{\alpha}V \Big)^{K} \xrightarrow{\mathrm{tr}_K^G} \Big(\bigoplus_{\substack{L \le G \\ \alpha : H \cong L}} {}^{\alpha}V \Big)^{G}.$$

If $\gamma : K \to G$ is any group homomorphism we define

$$J^*_{H,V}(\gamma) : \Big(\bigoplus_{\substack{L \leq G \\ \alpha : H \cong L}} {}^{\alpha}V \Big)^G \to \Big(\bigoplus_{\substack{L' \leq K \\ \beta : H \cong L'}} {}^{\beta}V \Big)^K$$

to have components $J^*_{H,V}(\gamma|_{L'}) : {}^{\alpha}V \to {}^{\beta}V$ if $\gamma(L') = L$, and 0 otherwise. In case K is a subgroup of G and γ is inclusion, this map is given by projection into those summands ${}^{\alpha}V$ for which the corresponding subgroup $L = \alpha(H)$ is a subgroup of K. In case $\gamma : K \to G$ is an group isomorphism then $J^*_{H,V}(\gamma)$ is determined on the summands ${}^{\alpha}V$ by the isomorphisms $\gamma|_L : L \to \gamma(L)$, in the way in which $J^*_{H,V}$ had previously been defined on isomorphisms. We define $(J_{H,V})_*(\gamma) = (J_{H,V})^*(\gamma)^{-1}$. By this means $(J_{H,V})_*$ is defined by functoriality on all monomorphisms of groups, since a monomorphism is always expressible as the composite of an isomorphism and an inclusion, and on these maps $(J_{H,V})_*$ has been defined.

This construction we have just described is really an extension of the one performed in [14, Sect. 4]. We refer to this paper for the verification of the Mackey formula (4′), and the fact that the other axioms for an inflation functor are satisfied is a straightforward check. We also see that $J_{H,V}$ is functorial in V, so we have a functor

$$\begin{aligned} \mathcal{G} : R[\mathrm{Out}(H)]\text{-mod} &\to \mathrm{Mack}^* \\ V &\mapsto J_{H,V} \end{aligned}$$

and also a functor

$$\mathcal{G} : R[\mathrm{Out}(H)]\text{-mod} \to \mathrm{Mack}$$

given by restricting $J_{H,V}$ to be a global Mackey functor. Notice from the definition that $J_{H,V}(H) = V$ and $J_{H,V}(K)$ vanishes unless K has a subgroup isomorphic to H.

(2.2) PROPOSITION.

(i) *The functor* $\mathcal{G} : V \mapsto J_{H,V} \in \mathrm{Mack}^*$ *is right adjoint to the functor* $\mathcal{F} : \mathrm{Mack}^* \to R[\mathrm{Out}(H)]$*-mod.*

(ii) *The functor* $\mathcal{G} : V \mapsto J_{H,V} \in \mathrm{Mack}$ *is right adjoint to the functor* $\mathcal{F} : \mathrm{Mack} \to R[\mathrm{Out}(H)]$*-mod.*

Proof. (i) We construct the unit η and counit ϵ of the adjunction and first we consider η. For each inflation functor M we define a natural transformation $\eta : M \to J_{H,\overline{M}(H)}$ whose effect at a group G is

$$\eta : M(G) \stackrel{(q_L R^G_L)}{\longrightarrow} J_{H,\overline{M}(H)}(G) = \Big(\bigoplus_{\substack{L \leq G \\ H \cong L}} \overline{M}(L) \Big)^G .$$

The notation here is that $q_L : M(L) \to \overline{M}(L)$ is the natural quotient, and so η is the map whose components are $q_L R_L^G$. We should verify that this is indeed a morphism of inflation functors, and we show that the two squares

$$
\begin{array}{ccc}
M(G) & \xrightarrow{(q_L R_L^G)} & \Big(\bigoplus_{\substack{L \le G \\ H \cong L}} \overline{M}(L)\Big)^G \\
{\scriptstyle M^*(\gamma)}\big\downarrow & & \big\downarrow {\scriptstyle J^*_{H,\overline{M}(H)}(\gamma)} \\
M(K) & \xrightarrow{(q_{L'} R_{L'}^K)} & \Big(\bigoplus_{\substack{L' \le K \\ H \cong L'}} \overline{M}(L')\Big)^K
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
M(G) & \xrightarrow{(q_L R_L^G)} & \Big(\bigoplus_{\substack{L \le G \\ H \cong L}} \overline{M}(L)\Big)^G \\
 & & \big\uparrow {\scriptstyle \mathrm{tr}_K^G} \\
{\scriptstyle I_K^G}\big\uparrow & & \Big(\bigoplus_{\substack{L \le G \\ H \cong L}} \overline{M}(L)\Big)^K \\
 & & \cup \\
M(K) & \xrightarrow{(q_L R_L^K)} & \Big(\bigoplus_{\substack{L \le K \\ H \cong L}} \overline{M}(L)\Big)^K
\end{array}
$$

commute, where $\gamma : K \to G$ is any group homomorphism in the first square, and K is any subgroup of G in the second square. Commutativity of the first square is immediate from the definition of $J^*_{H,\overline{M}(H)}(\gamma)$. The fact that the second square commutes is less obvious, but the argument is the same as the one which appears in Theorem 4.1 of [14]. We have

$$
\begin{aligned}
q_L R_L^G I_K^G &= q_L \Big(\sum_{g \in [L \backslash G / K]} I^L_{L \cap {}^gK} \, c_g \, R^K_{L^g \cap K} \Big) \\
&= \sum_{\substack{g \in [L \backslash G/K] \\ L \le {}^gK}} q_L \, c_g \, R^K_{L^g} \\
&= \sum_{\substack{g \in [G/K] \\ L \le {}^gK}} q_L \, c_g \, R^K_{L^g}
\end{aligned}
$$

since a double coset LgK satisfying $L \le {}^gK$ is equal to a coset gK, and using the fact that q_L kills all properly induced terms. This is the composition of morphisms round one side of the square, and the composition round the other way is

$$
\sum_{g \in [G/K]} c_g \sum_{\substack{L' \le K \\ L' \cong H}} q_{L'} \, R^K_{L'} = \sum_{\substack{g \in [G/K] \\ L' \le K}} q_{{}^gL'} \, c_g \, R^K_{L'}.
$$

The component of this with respect to a subgroup $H \cong L \le G$ is

$$
\sum_{\substack{g \in [G/K] \\ L = {}^gL' \le {}^gK}} q_L \, c_g \, R^K_{L^g}
$$

as required.

We now define the counit ϵ. At each $R[\mathrm{Out}(H)]$-module V, ϵ is simply the identity map $\epsilon : \overline{J_{H,V}}(H) \to V$. We next need to check that η and ϵ are natural transformations of functors

$$
\eta : 1 \to \mathcal{G}\mathcal{F}, \qquad \epsilon : \mathcal{F}\mathcal{G} \to 1.
$$

The case of ϵ is immediate since in fact $\mathcal{F}\mathcal{G}$ is the identity functor. As for η, we have to show that if $\theta : M \to N$ is a morphism of global Mackey functors then

$$\begin{array}{ccc} M(G) & \stackrel{(q_L R^G_L)}{\longrightarrow} & \left(\bigoplus_{H\cong L\leq G} \overline{M}(L)\right)^G \\ \theta_G \big\downarrow & & \big\downarrow (\overline{\theta}_L) \\ N(G) & \stackrel{(q_L R^G_L)}{\longrightarrow} & \left(\bigoplus_{H\cong L\leq G} \overline{N}(L)\right)^G \end{array}$$

commutes, where $\overline{\theta}_L$ is the morphism induced on $\overline{M}(L)$ by θ. This follows from the fact that θ commutes with the maps R^G_L.

We finally check that the two composites

$$\mathcal{F}\stackrel{\mathcal{F}\eta}{\longrightarrow}\mathcal{F}\mathcal{G}\mathcal{F}\stackrel{\epsilon_{\mathcal{F}}}{\longrightarrow}\mathcal{F} \qquad \mathcal{G}\stackrel{\eta_{\mathcal{G}}}{\longrightarrow}\mathcal{G}\mathcal{F}\mathcal{G}\stackrel{\mathcal{G}\epsilon}{\longrightarrow}\mathcal{G}$$

are the identity. This is easy, and we leave the details to the reader.

(ii) Exactly the same unit and counit demonstrate that we have an adjunction. □

(2.3) PROPOSITION. *Let V be a simple $R[\mathrm{Out}(H)]$-module.*

(i) As an inflation functor, every non-zero subfunctor of $J_{H,V}$ contains $J_{H,V}(H)$. Hence the subfunctor $\langle J_{H,V}(H)\rangle_{\mathrm{Mack}^*}$ generated by $J_{H,V}(H)$ in the category Mack* *is the unique minimal subfunctor of $J_{H,V}$, and hence is a simple inflation functor.*

(ii) As a global Mackey functor, every non-zero subfunctor of $J_{H,V}$ contains $J_{H,V}(H)$. Hence the subfunctor $\langle J_{H,V}(H)\rangle_{\mathrm{Mack}}$ generated by $J_{H,V}(H)$ in the category Mack *is the unique minimal subfunctor of $J_{H,V}$, and hence is a simple global Mackey functor.*

Proof. The proofs of (i) and (ii) are the same. Let $0 \neq M \subseteq J_{H,V}$ be a subfunctor. The inclusion map corresponds by adjointness to a non-zero homomorphism $\overline{M}(H) \to J_{H,V}(H) = V$ which must be an epimorphism since V is simple. Since $M(H) \subseteq V$ we must have $M(H) = V$ and so $M(H) \supseteq J_{H,V}(H)$. The final assertion is immediate. □

We define

$$\mathbb{S}_{H,V} = \langle J_{H,V}(H)\rangle_{\mathrm{Mack}^*}$$
$$\mathcal{S}_{H,V} = \langle J_{H,V}(H)\rangle_{\mathrm{Mack}}$$

to be the unique simple subfunctors of $J_{H,V}$, as an inflation functor, and as a global Mackey functor. Because of the structure of $J_{H,V}$ we may immediately say that H is the unique minimal group of both $\mathbb{S}_{H,V}$ and $\mathcal{S}_{H,V}$, and that $\mathbb{S}_{H,V}(H) = V = \mathcal{S}_{H,V}(H)$. We will go further than this and produce explicit formulae for both $\mathbb{S}_{H,V}(G)$ and $\mathcal{S}_{H,V}(G)$ which demonstrate that $\mathbb{S}_{H,V}$ and $\mathcal{S}_{H,V}$ are different functors, but first we show that we do have a complete list of the simple functors.

(2.4) THEOREM.

(i) *The simple inflation functors are precisely the* $\mathbb{S}_{H,V}$, *one for each pair* (H,V) *where* H *is a finite group and* V *is a simple* $R[\mathrm{Out}(H)]$*-module, both taken up to isomorphism.*

(ii) *The simple global Mackey functors are precisely the* $\mathcal{S}_{H,V}$, *one for each pair* (H,V) *where* H *is a finite group and* V *is a simple* $R[\mathrm{Out}(H)]$*-module, both taken up to isomorphism.*

Proof. We prove (ii), the proof of (i) being similar. We have shown that the $\mathcal{S}_{H,V}$ are simple, and it remains to show that these are all of the simple global Mackey functors. Let S be a simple global Mackey functor, H the unique minimal subgroup of S, and $V = S(H)$ the corresponding simple $R[\mathrm{Out}(H)]$-module. The isomorphism $S(H) = \overline{S}(H) \to V$ corresponds by adjointness to a non-zero map $S \to J_{H,V}$. Since S is simple this gives an isomorphism to the unique simple subfunctor, $S \cong \mathcal{S}_{H,V}$. □

We now embark on an explicit description of the functors we have constructed.

(2.5) LEMMA. *Let* $\mathcal{X}$ *be a class of groups closed under taking isomorphisms and taking subgroups.*

(i) *Let* M *be an inflation functor. Then the subfunctor* N *of* M *generated by the* $M(H)$ *with* $H \in \mathcal{X}$ *has the form*

$$N(G) = \sum_{G \overset{i}{\hookleftarrow} J \overset{\pi}{\to} H \in \mathcal{X}} i_*\pi^* M(H)$$

where the sum is taken over inclusions of subgroups i *and epimorphisms* π.

(ii) *Let* M *be a global Mackey functor. Then the subfunctor* N *of* M *generated by the* $M(H)$ *with* $H \in \mathcal{X}$ *has the form*

$$N(G) = \sum_{G \overset{i}{\hookleftarrow} H \in \mathcal{X}} i_* M(H)$$

where the sum is taken over inclusions of subgroups i.

Proof. (i) The right hand side is certainly contained in the left, and we have to show that the specification given by the right hand side does indeed give an inflation functor. To do this we show that the right hand side is stable under j_* and γ^* whenever $j : G \hookrightarrow G''$ is an inclusion of groups and $\gamma : G' \to G$ is any homomorphism of groups. Now $j_* i_* \pi^* M(H) = (ji)_* \pi^* M(H)$ shows stability under j_*. With γ it suffices to consider separately the cases when γ is an inclusion of groups and when γ is an epimorphism, since an arbitrary γ may be expressed as a composite of these. When γ is an inclusion $\gamma^* i_* \pi^*$ may be written as a sum of terms $i'_* \gamma'^* \pi^* = i'_*(\pi\gamma')^*$ using the Mackey formula, where now γ' is a monomorphism of the form $\gamma' : G' \cap {}^gJ \xrightarrow{c_g} G'^g \cap J \hookrightarrow G$. We now have $i'_*(\pi\gamma')^*M(H) \subseteq i'_*(\pi|_{G'^g \cap J}c_g)^*M(\pi(G'^g \cap J))$ and here $\pi(G'^g \cap J) \in \mathcal{X}$.

When γ is an epimorphism then

$$\gamma^* i_* \pi^* = i'_*(\gamma|_{J'})^* \pi^* = i'_*(\pi\gamma|_{J'})^*$$

where J' is the preimage of J in G' under γ and the maps are as follows:

$$\begin{array}{ccc} G' & \xrightarrow{\gamma} & G \\ \Big\uparrow{\scriptstyle i'} & & \Big\uparrow{\scriptstyle i} \\ J' & \xrightarrow{\gamma|_{J'}} & J \end{array}$$

This shows that our expression of $N(G)$ is stable under j_* and γ^* and hence gives an inflation functor.

(ii) The proof is similar to the proof of (i). We observe first that the right hand side is contained in the left. Then we show that the right hand side is stable under j_* and j^* whenever j is a monomorphism of groups. The arguments here are the same as those used to prove part (i), where the more general possibility of j being an arbitrary homomorphism was considered in the case of j^*. □

(2.6) THEOREM.

(i) The functors $J_{H,V}$ are given explicitly by

$$J_{H,V}(G) = \bigoplus_{\substack{\alpha : H \cong L \leq G \\ \text{up to } G-\text{conjugacy}}} ({}^\alpha V)^{N_G(L)}.$$

(ii) The simple global Mackey functors are given explicitly by

$$\mathcal{S}_{H,V}(G) = \bigoplus_{\substack{\alpha : H \cong L \leq G \\ \text{up to } G-\text{conjugacy}}} \operatorname{tr}_L^{N_G(L)}({}^\alpha V).$$

(iii) The simple inflation functors are given by

$$\mathbb{S}_{H,V}(G) = \sum_{\substack{\iota : K \hookrightarrow G \\ \pi : K \to H \text{ split epi}}} \iota_* \pi^*(V)$$

as a submodule of $J_{H,V}(G)$. For a given subgroup $K \leq G$ and a split epimorphism $\pi : K \to H$, if we write $\iota_*\pi^*(v) = \sum_{\substack{\alpha : H \cong L \leq G \\ \text{up to } G-\text{conjugacy}}} f_\alpha(v)$ as a sum of components with respect to the decomposition given in (i), where $v \in V$, then

$$f_\alpha(v) = \Big(\sum_{\substack{g \in [K \backslash T(L,K)] \\ B \cap {}^gL = 1}} \theta_g^{-1}\Big) \cdot v$$

where $T(L,K)=\{g\in G \mid {}^gL\subseteq K\}$, $B=\operatorname{Ker}\pi$ *and*

$$\theta_g : H\overset{\alpha}{\to}L\overset{c_g}{\to}{}^gL\hookrightarrow K\overset{\pi}{\to}H.$$

Note in (iii) that for a particular subgroup $L=\alpha(H)$ of G, if no G-conjugate of L is contained in K then $f_\alpha(v)=0$ since the sum which appears in the expression for this is empty. It may help in dealing with the notation here to observe that $\sum_{\substack{g\in[K\backslash T(L,K)]\\ B\cap {}^gL=1}}\theta_g^{-1}$ is an element of the group ring $R[\operatorname{Aut}(H)]$, which represents an element of $R[\operatorname{Out}(H)]$.

Proof. (i) The definition says that

$$J_{H,V}(G)=\Big(\bigoplus_{\alpha:H\cong L\leq G}{}^{\alpha}V\Big)^G.$$

Let us fix a particular subgroup $L\leq G$ isomorphic to H. The summands indexed by the conjugates of L form a summand

$$\begin{aligned}\Big(\bigoplus_{\substack{\alpha':H\cong L'\leq G\\ L'\sim_G L}}{}^{\alpha'}V\Big)^G &\cong ({}^{\alpha}V\uparrow_{N_G(L)}^G)^G\\ &\cong ({}^{\alpha}V)^{N_G(L)}.\end{aligned}$$

which gives the result. We obtain the first isomorphism because the term in parentheses on the left is the direct sum of subspaces permuted by G, of which one is ${}^{\alpha}V$ with stabilizer $N_G(L)$. The second isomorphism is a standard identification.

(ii) We have by 2.5(ii) that

$$\mathcal{S}_{H,V}(G)=\sum_{\alpha:H\cong L\leq G}I_L^G({}^{\alpha}V)$$

as a subset of $J_{H,V}(G)$, and it is a question of examining the maps I_L^G. They were defined as a composite which in this instance is

$${}^{\alpha}V\xrightarrow{1}{}^{\alpha}V\xrightarrow{\operatorname{tr}_L^G}\Big(\bigoplus_{\alpha':H\cong L'\leq G}{}^{\alpha'}V\Big)^G.$$

This has image contained in the summand determined by the conjugates of L, and identifying this with $({}^{\alpha}V\uparrow_{N_G(L)}^G)^G\cong {}^{\alpha}V^{N_G(L)}$ as in part (i), the composite map is $\operatorname{tr}_L^{N_G(L)}: {}^{\alpha}V\to {}^{\alpha}V^{N_G(L)}$. Hence $I_L^G({}^{\alpha}V)\cong \operatorname{tr}_L^{N_G(L)}\,{}^{\alpha}V$. We notice that if L is conjugate to L_1 in G then $I_{L_1}^G({}^{\alpha_1}V)=I_L^G({}^{\alpha}V)$ as subsets of $J_{H,V}(G)$, and if L,L_1 are subgroups of G which are isomorphic to H but are not conjugate in G then $I_{L_1}^G({}^{\alpha_1}V)$ and $I_L^G({}^{\alpha}V)$ are contained in different summands in the expression for $J_{H,V}(G)$. Thus our expression for $\mathcal{S}_{H,V}(G)$ is a direct sum taken over representatives of the G-conjugacy classes of subgroups L, and our identification of these summands gives the desired result.

(iii) To prove the first assertion we apply 2.5(i) with $\mathcal{X}$ consisting of all groups isomorphic to subgroups of H and $M = J_{H,V}$. Since $J_{H,V}(L) = 0$ for $L \in \mathcal{X}$ unless $L \cong H$ we have by that result that

$$\mathbb{S}_{H,V}(G) = \sum_{G \overset{i}{\hookleftarrow} K \overset{\pi}{\to} H} \iota_* \pi^*(V).$$

From the definition of the functor $J_{H,V}$ we have $\pi^* = J^*_{H,V}(\pi) = 0$ unless there is a subgroup $L \le K$ mapped isomorphically onto H by π. Thus if $J^*_{H,V}(\pi) \ne 0$ the group K must be a semidirect product $K = B \rtimes L$, and π is a split epimorphism.

We now examine the component f_α of the map $\iota_* \pi^*$. From the definition,

$$\pi^* : V \to \Big(\bigoplus_{\substack{L \le K \\ \alpha : H \cong L}} {}^\alpha V \Big)^K$$

is the map

$$\sum_{L' \text{ complements } B \text{ in } K} \pi|^*_{L'}.$$

On applying ι_*, only those complements to B in K which are G-conjugate to L contribute to the L summand of $J_{H,V}(G)$. The set of all G-conjugates of L in K has as a set of representatives the ${}^gL$ where $g \in [T(L,K)/N_G(L)]$ and so the component of π^* corresponding to the complements to B in K which are G-conjugate to L is

$$\sum_{\substack{g \in [T(L,K)/N_G(L)] \\ B \cap {}^gL = 1}} \pi|^*_{{}^gL}.$$

Let $k \in K$ and consider a complement ${}^{kg}L$ to B in K. The diagram

$$\begin{array}{ccc} {}^gL & \xrightarrow{\pi} & H \\ {\scriptstyle c_k}\downarrow & & \downarrow{\scriptstyle c_{\pi(k)}} \\ {}^{kg}L & \xrightarrow{\pi} & H \end{array}$$

commutes, and $c_{\pi(k)}$ acts trivially on V since it is an inner automorphism, so $(\pi|_{{}^{kg}L})^* = c_k \pi|^*_{{}^gL}$. Thus

$$\begin{aligned} \sum_{\substack{g \in [T(L,K)/N_G(L)] \\ B \cap {}^gL = 1}} \pi|^*_{{}^gL} &= \sum_{\substack{g \in [K \backslash T(L,K)/N_G(L)] \\ B \cap {}^gL = 1}} \sum_{k \in [K/K \cap {}^gN_G(L)]} \pi|^*_{{}^{kg}L} \\ &= \sum_{\substack{g \in [K \backslash T(L,K)/N_G(L)] \\ B \cap {}^gL = 1}} \operatorname{tr}^K_{K \cap {}^gN_G(L)} \pi|^*_{{}^gL}. \end{aligned}$$

Now the L component in $J_{H,V}(G)$ of $\iota_*\pi^*$ is

$$\begin{aligned}\mathrm{tr}_K^G \sum_{\substack{g\in[K\backslash T(L,K)/N_G(L)]\\ B\cap {}^gL=1}} \mathrm{tr}^K_{K\cap {}^gN_G(L)}\,\pi|^*_{{}^gL} &= \sum_{\substack{g\in[K\backslash T(L,K)/N_G(L)]\\ B\cap {}^gL=1}} \mathrm{tr}^G_{K\cap {}^gN_G(L)}\,\pi|^*_{{}^gL}\\ &= \sum_{\substack{g\in[K\backslash T(L,K)/N_G(L)]\\ B\cap {}^gL=1}} \mathrm{tr}^G_{K^g\cap N_G(L)}\,c_{g^{-1}}\pi|^*_{{}^gL}\\ &= \mathrm{tr}^G_{N_G(L)} \sum_{\substack{g\in[K\backslash T(L,K)/N_G(L)]\\ B\cap {}^gL=1}} \mathrm{tr}^{N_G(L)}_{K^g\cap N_G(L)}\,c_{g^{-1}}\pi|^*_{{}^gL}\\ &= \mathrm{tr}^G_{N_G(L)} \sum_{\substack{g\in[K\backslash T(L,K)]\\ B\cap {}^gL=1}} c_{g^{-1}}\pi|^*_{{}^gL}\\ &\in \Big(\bigoplus_{\substack{L\le G\\ \alpha:H\cong L}} {}^{\alpha}V\Big)^G.\end{aligned}$$

Finally the isomorphism

$$\bigoplus_{\substack{H\overset{\alpha}{\cong}L\le G\\ \text{up to }G-\text{conjugacy}}} {}^{\alpha}V^{N_G(L)} \to \Big(\bigoplus_{\substack{L\le G\\ \alpha:H\cong L}} {}^{\alpha}V\Big)^G$$

on the L component is just $v \mapsto \mathrm{tr}^G_{N_G(L)}\,v$ and so composing with the inverse of this isomorphism we obtain that the L component f_α of $\iota_*\pi^*$ is

$$\sum_{\substack{g\in[K\backslash T(L,K)]\\ B\cap {}^gL=1}} c_{g^{-1}}\pi|^*_{{}^gL}.$$

Now $c_{g^{-1}}\pi|^*_{{}^gL}$ here denotes the homomorphism $V \to {}^{\alpha}V$ obtained by contravariant functoriality from the group homomorphism $L\xrightarrow{c_g}{}^gL \hookrightarrow K\xrightarrow{\pi}H$ using the isomorphism $\alpha : H \to L$. The functorial definition gives precisely $c_{g^{-1}}\pi|^*_{{}^gL}(v) = \theta_g^{-1}\cdot v$ where $v \in V$ and θ_g is defined in the statement of this theorem. Thus the component f_α of $\iota_*\pi^*$ is as claimed. □

(2.7) COROLLARY.

(i) $\mathcal{S}_{H,V}(G) \neq 0$ *if and only if there is a subgroup* $L \le G$ *isomorphic to* H *via* $\alpha : H \to L$ *so that* $\mathrm{tr}_L^{N_G(L)}({}^{\alpha}V) \neq 0$.

(ii) $\mathbb{S}_{H,V}(G) \neq 0$ *if and only if there are subgroups* L, K *and* B *of* G *with* L *isomorphic to* H *via* α *and* $K = B \rtimes L$ *so that* $\sum_{\substack{g\in[K\backslash T(L,K)]\\ B\cap {}^gL=1}} \theta_g^{-1}$ *does not annihilate* V.

We see from this that it is much easier to compute the global Mackey functors $\mathcal{S}_{H,V}(G)$ than the inflation functors $\mathbb{S}_{H,V}(G)$, and in Section 4 we will compute a number of values of $\mathcal{S}_{H,V}(G)$. It is entirely by reason of this ease of computation that we use global Mackey functors in our application to the computation of cohomology, although in principle the computations could be done with inflation functors as well. The description of $\mathbb{S}_{H,V}$ which we have just given is, however, exactly what we need to reprove the results of Martino-Priddy [8] and Benson-Feshbach [1] concerning decomposition of classifying spaces, which we will do in Section 6.

One may give more straightforward sufficient conditions for when these simple functors vanish. The result in the case of inflation functors is really the same as a condition which has been obtained by Benson and Feshbach [1, Proposition 5.9] in an apparently different context. We will show later that their situation really amounts to the same thing as ours, and our result can be deduced from theirs by translating the concepts.

Let L be a subgroup of G and let W be a $R[\mathrm{Out}(L)]$-module. Then W becomes a $RN_G(L)$-module via the homomorphisms $N_G(L) \to \mathrm{Aut}(L) \to \mathrm{Out}(L)$. We define $\mathrm{Stab}_G(W)$ to be the kernel of this action of $N_G(L)$ on W. Thus

$$\mathrm{Stab}_G(W) = \{g \in N_G(L) \mid g \cdot w = w \text{ for all } w \in W\}.$$

Since $C_G(L)$ always acts trivially on L by conjugation we have $L \cdot C_G(L) \leq \mathrm{Stab}_G(W)$.

(2.8) PROPOSITION. *Let G and H be p-groups, let R be a field of characteristic p and let V be a simple $R[\mathrm{Out}(H)]$-module.*

(i) If $\mathcal{S}_{H,V}(G) \neq 0$ then there exists a subgroup $L \leq G$ with an isomorphism $\alpha : H \to L$ such that $\mathrm{Stab}_G({}^{\alpha}V) = L$ (i.e. $N_G(L)/L$ acts faithfully on ${}^{\alpha}V$). In particular $C_G(L) \leq L$.

(ii) If $\mathbb{S}_{H,V}(G) \neq 0$ then there exists a subgroup $L \leq G$ with an isomorphism $\alpha : H \to L$ such that $\mathrm{Stab}_G({}^{\alpha}V) = C_G(L) \cdot L = L \times X$ for some subgroup X of G.

Proof. (i) From 2.6(ii) we deduce that if $\mathcal{S}_{H,V}(G) \neq 0$ then

$$\mathrm{tr}_L^{N_G(L)} = \mathrm{tr}_{\mathrm{Stab}_G({}^{\alpha}V)}^{N_G(L)} \mathrm{tr}_L^{\mathrm{Stab}_G({}^{\alpha}V)} \neq 0$$

for some subgroup L. Hence $\mathrm{tr}_L^{\mathrm{Stab}_G({}^{\alpha}V)} \neq 0$ and since G is a p-group it follows that $\mathrm{Stab}_G({}^{\alpha}V) = L$.

(ii) Using notation introduced in 2.6(iii) we have

$$\mathbb{S}_{H,V}(G) = \sum_{\substack{\iota: K \hookrightarrow G \\ \pi: K \to H \text{ split epi}}} (\sum_{\alpha} f_\alpha)V,$$

for this expression to be non-zero there must be a subgroup K and an isomorphism $\alpha : H \to L$ for some subgroup L of G such that $f_\alpha \neq 0$. Now

$$f_\alpha = \sum_{\substack{g \in [K \backslash T(L,K)/N_G(L)] \\ B \cap {}^{g}L = 1}} \mathrm{tr}_{N_{K^g}(L)}^{N_G(L)} c_{g^{-1}} \pi|^*_{{}^{g}L}$$

and so $\mathrm{tr}_{N_{K^g}(L)}^{N_G(L)}$ must be non-zero for some g. Now K^g is a semidirect product $K^g = B^g \rtimes L$ and so $N_{K^g}(L) = X \times L$ where $X = C_{B^g}(L)$. Thus $N_{K^g}(L) \leq C_G(L) \cdot L \leq \mathrm{Stab}_G({}^\alpha V)$ and since

$$\mathrm{tr}_{N_{K^g}(L)}^{N_G(L)} = \mathrm{tr}_{\mathrm{Stab}_G({}^\alpha V)}^{N_G(L)} \mathrm{tr}_{N_{K^g}(L)}^{\mathrm{Stab}_G({}^\alpha V)}$$

it follows that $\mathrm{tr}_{N_{K^g}(L)}^{\mathrm{Stab}_G({}^\alpha V)} \neq 0$. Because G is a p-group and $\mathrm{Stab}_G({}^\alpha V)$ acts trivially on ${}^\alpha V$ we deduce that $N_{K^g}(L) = \mathrm{Stab}_G({}^\alpha V)$, which in turn must coincide with $X \times L$ and $C_G(L) \cdot L$. □

(2.9) COROLLARY. *Let G and H be p-groups, let R be a field of characteristic p, and let R also denote the trivial $R[\mathrm{Out}(H)]$-module.*
(i) $\mathcal{S}_{H,R}(G) \neq 0$ if and only if $H \cong G$.
(ii) If $\mathbb{S}_{H,R}(G) \neq 0$ then there is a subgroup L of G isomorphic to H for which $N_G(L) = L \times X$ for some subgroup X.

Proof. We have $\mathrm{Stab}_G({}^\alpha V) = N_G(L)$ for every subgroup L of G isomorphic to H, and (ii) is immediate from 2.8(ii). In case (i), if $\mathcal{S}_{H,R}(G) \neq 0$ we have by 2.8(i) that $N_G(L) = L$, which forces $L = G$. Conversely, if $H \cong G$ we have $\mathcal{S}_{H,R}(G) = R$, as remarked before 2.4. □

3. The composition factors of a global Mackey functor

In this section we consider only global Mackey functors, because that is the context in which we will apply the theory about to be developed. A similar theory may be developed for inflation functors, and we leave this to the interested reader.

We will be dealing with global Mackey functors which do not have composition series in the conventional sense and so we need to give some meaning to the notion of composition factor for a global Mackey functor. To begin, we will say that a simple global Mackey functor S is a composition factor of M if there exist subfunctors $M_0 \subset M_1 \subseteq M$ so that $M_1/M_0 \cong S$. To extend this notion, we fix a group G and say that a global Mackey functor has a *composition series over* G if there is a series of subfunctors

$$0 = T_0 \subseteq B_1 \subset T_1 \subseteq \cdots \subseteq B_m \subset T_m \subseteq B_{m+1} = M$$

such that

T_i/B_i is simple and non-zero on restriction to G for all $i = 1, \ldots, m$, and

$(B_{i+1}/T_i)\downarrow_G = 0$ for all $i = 0, \ldots, m$.

When such a series exists we will call the set of simple functors T_i/B_i together with their multiplicities, the *composition factors of M over G.* We are thus pulling out just those composition factors of M which do not vanish on G.

(3.1) PROPOSITION. *Suppose that M is a global Mackey functor which has a composition series over G. Then any other composition series over G for M has the same length and the composition factors over G (taken with multiplicities) are the same.*

Proof. By a standard argument using the Zassenhaus isomorphism theorem, any two series

$$0 \subseteq B_1 \subset T_1 \subseteq \cdots \subseteq B_m \subset T_m \subseteq M$$

have refinements which are of equal length and in which the sets of quotients are the same (up to isomorphism, taken with multiplicities). It is impossible to refine the simple quotients T_i/B_i any further, and so in any refinement of the above series, the set of quotients which do not vanish on G remains the same. From this we see that in any other composition series for M over G the composition factors over G are the same, and in particular there is the same number of them. □

We are now concerned with the existence of composition series over G, and for this we will use a characterisation of simple global Mackey functors. Let $\mathcal{X}$ be a class of groups closed under isomorphism and taking subgroups. If M is a global Mackey functor we introduce the notation $\operatorname{Ker} R_{\mathcal{X}}$ to denote the subfunctor of M defined by

$$(\operatorname{Ker} R_{\mathcal{X}})(K) = \bigcap_{\substack{X \in \mathcal{X} \\ X \leq K}} \operatorname{Ker} R^K_X.$$

(3.2) THEOREM. *Let S be a global Mackey functor, let H be a minimal group for S and let $\mathcal{X}$ be the class of groups isomorphic to subgroups of H. Then S is simple if and only if the following three conditions are satisfied.*

(i) $\operatorname{Ker} R_{\mathcal{X}} = 0$,

(ii) $S = \langle S(H) \rangle$,

(iii) $S(H)$ *is a simple* $R[\operatorname{Out}(H)]$*-module.*

Proof. Assume S is simple. Since $(\operatorname{Ker} R_{\mathcal{X}})(H) = 0$, $\operatorname{Ker} R_{\mathcal{X}}$ is a proper subfunctor, hence is zero. The other two conditions have already been established.

Assume now that the three conditions hold and let T be a non-zero subfunctor of S. Let K be a group for which $T(K) \neq 0$. By (i)

$$(\operatorname{Ker} R_{\mathcal{X}})(K) = \bigcap_{H \cong L \leq K} \operatorname{Ker} R^K_L = 0$$

and so there exists a subgroup $L \leq K$ isomorphic to H for which $T(K) \not\subseteq \operatorname{Ker} R^K_L$. Thus $R^K_L(T(K)) \neq 0$ and so $T(L) \neq 0$ and hence $S(H) \supseteq T(H) \neq 0$. Since $S(H)$ is a simple module we have $S(H) = T(H)$. Finally we deduce $S = T$ from (ii). □

(3.3) THEOREM. *Let G be a finite group and M a global Mackey functor over R. The following are equivalent:*

(i) M has a composition series over G,

(ii) for all subgroups $K \le G$, $M(K)$ has a composition series as an R-module,

(iii) $M\downarrow_G$ has a composition series as a Mackey functor for G.

Proof. The implication (i) $\Rightarrow$ (ii) is easy, since the restriction of a composition series over G to K gives a filtration of $M(K)$ in which the finitely many successive quotients are evaluations of simple global Mackey functors $\mathcal{S}_{H,V}(K)$. From our description of the simple functors, these all have finite length over some field which is a quotient of R.

For the converse implication (ii) $\Rightarrow$ (i), we proceed by induction on

$$\ell(M) = \sum_{K \le G} \mathrm{length}_R(M(K)),$$

the induction starting when this number is 0. We suppose, therefore, that $\ell(M) > 0$ and that the result holds for global Mackey functors with smaller values of ℓ. Let $\mathcal{Y}$ consist of all groups isomorphic to subgroups of G and let $B_1 = \mathrm{Ker}\, R_{\mathcal{Y}}$ as a subfunctor of M. Then $B_1\downarrow_G = 0$, and we show that the quotient M/B_1 has a simple subfunctor which is a composition factor over G. To construct this simple subfunctor, let $J \le G$ be a subgroup of maximal order such that

$$\bigcap_{K \lneqq J} \mathrm{Ker}\, R^J_K \ne 0$$

and let $0 \ne V \le M(J)$ be a simple $R[\mathrm{Out}(J)]$-submodule of the above intersection. Since every element of V is sent to 0 on restriction to all proper subgroups of J, the subfunctor of M generated by V has the form

$$\langle V\rangle(K) = \sum_{J \cong J_1 \le K} \mathrm{Im}(I^K_{J_1})$$

and in particular $\langle V\rangle$ vanishes unless K contains a subgroup isomorphic to J. We put $T_1 = B_1 + \langle V\rangle$, and show that T_1/B_1 is simple by verifying the three conditions in our characterisation of simple global Mackey functors.

Since J is a minimal group for $\langle V\rangle$ and B_1 vanishes on subgroups of G, we see that J is also a minimal group for T_1/B_1, and furthermore $(T_1/B_1)(J) = \langle V\rangle(J) = V$ is a simple $R[\mathrm{Out}(J)]$-module. It is clear that T_1/B_1 is generated by its value at J. Thus it remains to show that if $\mathcal{X}$ denotes the class of groups isomorphic to subgroups of J then $\mathrm{Ker}\, R_{\mathcal{X}} = 0$ for the functor T_1/B_1.

Consider an element $x + B_1(K) \in \mathrm{Ker}\, R_{\mathcal{X}}(K)$ for some subgroup K, where $x \in \langle V\rangle(K)$. We show that $x \in B_1(K)$, which will complete our argument that T_1/B_1 is simple. Suppose to the contrary that $x \notin B_1(K)$, so there exists $L \in \mathcal{Y}$ with $R^K_L(x) \ne 0$. Since $\langle V\rangle$ vanishes on groups which do not contain a subgroup isomorphic to J, L contains such a subgroup. We show that for some subgroup J_1 of L isomorphic to J, $R^K_{J_1}(x) \neq 0$. The argument is by induction on $|L|$. If $|L| = |J|$ then $L \cong J$ and $R^K_L(x) \neq 0$ already, so the induction starts. Supposing now that $|L| > |J|$ and the result holds for smaller values of $|L|$, by maximality of $|J|$ there exists a proper subgroup L_1 of L with $R^K_{L_1}(x) = R^L_{L_1} R^K_L(x) \neq 0$, and necessarily L_1 still contains a subgroup isomorphic to J. By induction we deduce that $R^{L_1}_{J_1} R^K_{L_1}(x) = R^K_{J_1}(x) \neq 0$ for some subgroup $J_1 \leq L$, isomorphic to J. This contradicts the earlier hypothesis that $x + B_1(K) \in \operatorname{Ker} R_{\mathcal{X}}(K)$, and so we deduce that $x \in B_1(K)$, which shows that T_1/B_1 is simple.

We have now constructed the start of a chain $B_1 \subset T_1 \subseteq M$ where T_1/B_1 is a composition factor over G and $B_1 \downarrow_G = 0$. Now $\ell(M/T_1) < \ell(M)$, so by induction M/T_1 has a composition series over G, and we deduce that M does also.

The equivalence of (ii) and (iii) follows from [16]. □

As a way of recording the composition factors of a global Mackey functor we introduce an abelian group Π which is the product of copies of $\mathbb{Z}$ indexed by the simple global Mackey functors. Thus an element of Π is a family of integers $(n_{H,V})$ indexed by pairs (H, V), which on linearly ordering the pairs (H, V) we may write as a sequence. Suppose that M is a global Mackey functor with the property that for every finite group G, $M(G)$ has a composition series as an R-module. In this situation we may associate the element $[M] \in \Pi$, which is defined to be the sequence whose (H, V) term $[M]_{H,V}$ is the multiplicity of $\mathcal{S}_{H,V}$ in a composition series for M over H. This is also the multiplicity of $\mathcal{S}_{H,V}$ in a composition series for M over G, where G is any group containing a subgroup isomorphic to H. Thus the composition factors of M over G are precisely the $\mathcal{S}_{H,V}$ with multiplicity $[M]_{H,V}$, ranging over groups H which are isomorphic to subgroups of G. This means that the composition factors of M are determined by knowing $[M]$.

At this point we will assume that R is an artinian ring and form the product

$$\Pi' = \prod_H G_0(R[\operatorname{Out}(H)])$$

of the Grothendieck groups of $R[\operatorname{Out}(H)]$-modules taken over all finite groups H (up to isomorphism). A typical element may be written (a_H), where for each finite group H, $a_H \in G_0(k[\operatorname{Out}(H)])$. Because of our hypothesis on R, each Grothendieck group has a $\mathbb{Z}$-basis consisting of the simple $R[\operatorname{Out}(H)]$-modules and so we may write

$$a_H = \sum_V n_{H,V}[V]$$

where V ranges over the simple $R[\operatorname{Out}(H)]$-modules. Thus (a_H) may be represented by a family of integers $(n_{H,V})$ indexed by pairs (H, V), and since the elements of the group Π defined previously may be represented in the same way, we see that as abstract groups $\Pi \cong \Pi'$.

We define a homomorphism $\psi : \Pi \to \Pi'$ by

$$\psi((n_{H,V})) = (\sum_{(H,V)} n_{H,V}[\mathcal{S}_{H,V}(K)])$$

the expression on the right being a family of terms indexed by groups K, and where the term in parentheses on the right is an element of $G_0(R[\mathrm{Out}(K)])$. The sum here is apparently infinite since it is taken over all pairs (H,V), but in fact there are only finitely many terms in the sum for each K, since only those groups H which are isomorphic to subgroups of K contribute.

(3.4) LEMMA. *The morphism ψ is a monomorphism.*

Proof. Suppose that the indexing pairs (H,V) are linearly ordered in any fashion so that $|H|$ is non-decreasing, and that the groups which index the factors in Π' have the induced linear order. Let $a = (a_{H,V})$ and $b = (b_{H,V})$ be distinct sequences of integers such that $\psi(a) = \psi(b)$. Write $a = c + a_1$, $b = c + b_1$ where c is the longest common initial subsequence of both a and b, and a_1, b_1 are different in their first non-zero place. So

$$a_1 = (0, \ldots, 0, a_{H,V}, \ldots)$$
$$b_1 = (0, \ldots, 0, b_{H,V}, \ldots)$$

with $a_{H,V} \neq b_{H,V}$ and $\psi(a_1) = \psi(a) - \psi(c) = \psi(b) - \psi(c) = \psi(b_1)$. Since $\mathcal{S}_{H,V}(H) = V$ the definition of ψ gives that

$$\psi(0, \ldots, 0, a_{H,V}, \ldots) = (0, \ldots, 0, a_{H,V}[V] + \ldots, \ldots),$$

so that the coefficient of $[V]$ in the first non-zero entry is $a_{H,V}$, and similarly with $\psi(b_1)$. We thus deduce that $\psi(a_1)$ cannot equal $\psi(b_1)$, a contradiction, and so ψ is a monomorphism. □

As a consequence of this lemma, in order to check that a global Mackey functor M has composition factors given by the sequence $(n_{H,V})$, we only need check that $\psi([M]) = \psi((n_{H,V}))$. The advantage of this is that the Kth component $\psi([M])_K$ is just $[M(K)]$, and so we simply have to determine the module $M(K)$ as an element of $G_0(R[\mathrm{Out}(K)])$. Notice that in order to compute a composition factor multiplicity $n_{H,V}$ it suffices to compute the values of $[M(K)]$ where K is isomorphic to a subgroup of H. This results from the fact that the restriction of ψ to the product of those factors indexed by such groups K is again a monomorphism to the product of the corresponding factors in Π', by exactly the same argument as in the lemma.

We may exercise some further control over the groups H which appear in the parameterisation of the composition factors of a global Mackey functor M using essentially the relative projectivity as Mackey functors of the restrictions $M\downarrow_G$. For this we assume familiarity with the classification of the simple Mackey functors $S_{H,V}$ given in [15].

(3.5) PROPOSITION. *Let M be a global Mackey functor and $\mathcal{X}$ a class of groups closed under isomorphisms. Suppose that for every finite group G, the Mackey functor composition factors of $M\downarrow_G$ are all of the form $S_{H,V}$ with $H \in \mathcal{X}$. Then every composition factor of M as a global Mackey functor has the form $\mathcal{S}_{H,V}$ with $H \in \mathcal{X}$.*

Proof. Let $\mathcal{S}_{H,V}$ be a composition factor of M as a global Mackey functor. Thus there are subfunctors $M_0 \subset M_1 \subseteq M$ with $M_1/M_0 \cong \mathcal{S}_{H,V}$. On restricting this filtration of M to H we obtain Mackey functors $M_0\downarrow_H \subset M_1\downarrow_H \subseteq M\downarrow_H$, and now

$$(M_1\downarrow_H)/(M_0\downarrow_H) = \mathcal{S}_{H,V}\downarrow_H$$

is a Mackey functor which is non-zero only on H. From the structure of the simple Mackey functors [15] we deduce that the Mackey functor composition factors of $(M_1\downarrow_H)/(M_0\downarrow_H)$ must have the form $S_{H,W}$, and these are also composition factors of $M\downarrow_H$. By hypothesis we have $H \in \mathcal{X}$. □

4. The computation of global Mackey functors

We will now show how to use the notions of composition factor and composition series to compute the values of specific Mackey functors. In particular we will take as examples the global Mackey functors $H^r(G,T)$ where the coefficient module T has trivial G-action. In order that the Grothendieck groups $G_0(R[\mathrm{Out}(H)])$ should have a basis consisting of the simple modules we will assume in this section that the ring R over which our global Mackey functors are defined is an artinian ring. However, as far as the cohomology groups $H^r(G,T)$ with $r \geq 1$ are concerned we need not require that T itself be artinian. This is because $H^r(G,T)$ is annihilated by $|G|$ and so the cohomology may turn out to be a finitely generated module for an artinian ring, even though T is not. We have in mind the example of $H^r(G,\mathbb{Z})$, which is a finitely generated module for the artinian ring $R = \mathbb{Z}/|G|\mathbb{Z}$.

In general our method proceeds as follows. We wish to compute a certain value $M(G)$ of a global Mackey functor, and suppose that M has a composition series over G. Thus we have a filtration

$$0 = T_0 \subseteq B_1 \subset T_1 \subseteq \cdots \subseteq B_m \subset T_m \subseteq B_{m+1} = M$$

where each T_i/B_i is a simple composition factor over G and B_{i+1}/T_i vanishes on G. On evaluation at G this gives a filtration

$$0 = B_1(G) \subseteq T_1(G) = B_2(G) \cdots \subseteq T_{m-1}(G) = B_m(G) \subseteq T_m(G) = M(G)$$

in which there are finitely many quotients which are evaluations of simple functors $\mathcal{S}_{H,V}(G)$. We may conclude that if $\mathcal{S}_{H,V}$ occurs with multiplicity $n_{H,V}$ as a composition factor of M then

$$M(G) = \sum_{H,V} n_{H,V} \mathcal{S}_{H,V}(G) \in G_0(R[\mathrm{Out}(G)]),$$

the sum being taken over pairs (H, V) with H isomorphic to a subgroup of G. By determining the multiplicies $n_{H,V}$ using the morphism ψ of the last section and using the explicit formula 2.6(ii) for $\mathcal{S}_{H,V}(G)$ we obtain an expression for $M(G)$ in the Grothendieck group.

The success of this method depends on how easily these quantities may be computed, and since the explicit formula for $\mathcal{S}_{H,V}(G)$ is not so bad, the difficulty rests primarily with the $n_{H,V}$. To simplify matters in computing the $n_{H,V}$ we will try to find a class $\mathcal{X}$ of groups to which Proposition 3.5 applies. Since it is only necessary to compute $n_{H,V}$ in case $\mathcal{S}_{H,V}(G) \neq 0$ we look for conditions for this to happen, and use 2.8 and 2.9 in conjunction with the following lemma.

(4.1) LEMMA. *Let G be any group, H a p-group and P a Sylow p-subgroup of G. If $\mathcal{S}_{H,V}(P) = 0$ then $\mathcal{S}_{H,V}(G) = 0$.*

Proof. By the formula for simple global Mackey functors we have

$$\mathcal{S}_{H,V}(P) = \bigoplus_{H \overset{\alpha}{\cong} L \leq P} \mathrm{tr}_L^{N_P(L)}({}^{\alpha}V) = 0$$

so $\mathrm{tr}_L^{N_P(L)}({}^{\alpha}V) = 0$ for all such L. Hence $\mathrm{tr}_L^{N_G(L)}({}^{\alpha}V) = 0$ for all $H \overset{\alpha}{\cong} L \leq G$ since $\mathrm{tr}_L^{N_G(L)} = \mathrm{tr}_{N_P(L)}^{N_G(L)} \cdot \mathrm{tr}_L^{N_P(L)}$. So

$$\mathcal{S}_{H,V}(G) = \bigoplus_{H \overset{\alpha}{\cong} L \leq G} \mathrm{tr}_L^{N_G(L)}({}^{\alpha}V) = 0.$$

□

We turn to the specific case of group cohomology with finitely generated trivial coefficients $H^r(G, T)$. Since this cohomology group is a finite abelian group we may perform the calculation one prime at a time, computing the Sylow p-subgroup $H^r(G, T)_p$. Let us write $M^r(G) = H^r(G, T)_p$. It is well-known that on every group the Mackey functor which this defines is projective relative to p-subgroups, and in fact the Mackey functor composition factors of $M^r \downarrow_G$ are of the form $S_{H,V}$ where H is a p-group [16]. Thus Proposition 3.5 applies on taking $\mathcal{X}$ to be all p-groups, and we deduce that the composition factors of M^r as a global Mackey functor have the form $\mathcal{S}_{H,V}$ with H a p-group. Here V must necessarily be a simple module defined over a field of characteristic p which is a homomorphic image of our ground ring R

We may represent the map ψ considered in Section 3 by means of an infinite matrix Ψ with rows indexed by the finite groups (taken up to isomorphism), columns indexed by the simple global Mackey functors and entries in the Grothendieck groups $G_0(R[\mathrm{Out}(G)])$. The entry in row G and column $\mathcal{S}_{H,V}$ is the module $\mathcal{S}_{H,V}(G)$. Using 2.8 and 2.9 we can immediately say that many entries of Ψ are zero, and we now present part of Ψ in the case when R is any field of characteristic 2.

	1	C_2	C_4	$C_2 \times C_2$		C_8	Q_8		D_{2^n}	Q_{2^n}	SD_{2^n}
	1	1	1	1	2	1	1	2	1	1	1
1	1										
C_2		1									
C_4			1								
$C_2 \times C_2$				1	2						
C_8						1					
Q_8							1	2			
D_{2^n}					$1 \oplus 1$				1		
Q_{2^n}								$1 \oplus 1$		1	
SD_{2^n}					1			1			1

Most of the entries in this table are computed to be zero by Lemma 2.9. The only remaining entries in question are $\mathcal{S}_{C_2 \times C_2, 2}(P)$ and $\mathcal{S}_{Q_8,2}(P)$, which are computed using 2.6. In this table and throughout the ensuing calculation we denote the trivial module R by 1, and the natural 2-dimensional $GL(2,2){=}S_3$-module by 2.

We now explain how to use this table to compute the value of a global Mackey functor, illustrating the method by considering the cohomology $M^r(G) = H^r(G, \mathbb{F}_2)$ of a group G with D_{2^n} as a Sylow 2-subgroup, $n \geq 3$. By 4.1 we see from the table that

$$M^r(G) = n_{C_2 \times C_2, 2} \mathcal{S}_{C_2 \times C_2, 2}(G) + n_{D_{2^n}, 1} \mathcal{S}_{D_{2^n}, 1}(G) \in G_0(\mathbb{F}_2[\mathrm{Out}(G)]).$$

The first of these coefficients $n_{C_2 \times C_2, 2}$ is computed from the similarly obtained equation

$$M^r(C_2 \times C_2) = n_{C_2 \times C_2, 1} \mathcal{S}_{C_2 \times C_2, 1}(C_2 \times C_2) + n_{C_2 \times C_2, 2} \mathcal{S}_{C_2 \times C_2, 2}(C_2 \times C_2). \tag{4.2}$$

Here we see that $n_{C_2 \times C_2, 2}$ equals the multiplicity of the 2-dimensional simple $\mathbb{F}_2[\mathrm{Out}(C_2 \times C_2)]$-module as a composition factor of $M^r(C_2 \times C_2)$, so that a knowledge of the cohomology of $C_2 \times C_2$ gives this coefficient. Returning to the first equation, knowledge of $M^r(G)$ for any group G with Sylow 2-subgroup D_{2^n} will now suffice to determine $n_{D_{2^n}, 1}$ — for example, $G = D_{2^n}$ will do. In general we have to make the requirement that $\mathcal{S}_{D_{2^n}, 1}(G) \neq 0$ at this step, a requirement which in this particular case is always satisfied. Having determined the two coefficients we calculate $M^r(G)$ using the formula 2.6 for the simple global Mackey functors.

We may summarise the approach in general as follows: for a p-group P, $n_{P,V}$ can be computed knowing the values of $n_{H,W}$ for subgroups $H \leq P$ with $\mathcal{S}_{H,W}(P) \neq 0$ and knowing the cohomology of any group G which has P as a Sylow p-subgroup, for which $\mathcal{S}_{P,V}(G) \neq 0$. One may always take G to be P itself, but sometimes it may be more convenient to compute the cohomology of a bigger group G. Altogether one would have used as input a knowledge of the cohomology of a group with H as a Sylow p-subgroup, for every p-subgroup H under consideration. Finally, having determined the numbers $n_{H,V}$ in this way we obtain the cohomology of any group which has P as a Sylow p-subgroup by evaluating the relevant simple global Mackey functors at that group, using 2.6.

We now use this method to compute the additive structure of the cohomology of various groups at the prime 2. It is convenient to work with the Poincaré series

$$P_G(t) = \sum_{r=0}^{\infty} t^r \dim H^r(G, \mathbb{F}_2),$$

and we will use the notation $A_G(L) = N_G(L)/C_G(L)$ to denote the *automizer* of L in G. The cohomology of the groups specified in the next theorem has previously been determined in [7] and [9] by different methods.

(4.3) THEOREM.

(a) *Let G be a group with D_{2^n} as a Sylow 2-subgroup, $n \geq 3$. Then*

$$P_G(t) = \frac{1+t^3}{(1-t^2)(1-t^3)} + \frac{\lambda t}{(1-t)(1-t^3)}$$

where λ is the number of conjugacy classes of subgroups $L \cong C_2 \times C_2$ in G for which $3 \nmid |A_G(L)|$.

(b) *Let G be a group with SD_{2^n} as a Sylow 2-subgroup, $n \geq 4$. Then*

$$P_G(t) = \frac{1+t^5}{(1-t^3)(1-t^4)} + \frac{\lambda t(1+t)}{(1-t^4)} + \frac{\mu t}{(1-t)(1-t^3)}$$

where

$$\lambda = \begin{cases} 0 & \text{if } 3 \mid |A_G(Q_8)| \\ 1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \mu = \begin{cases} 0 & \text{if } 3 \mid |A_G(C_2 \times C_2)| \\ 1 & \text{otherwise} \end{cases}.$$

(c) *Let G be a group with Q_{2^n} as a Sylow 2-subgroup, $n \geq 3$. Then*

$$P_G(t) = \frac{1+t^3+\lambda t(1+t)}{(1-t^4)}$$

where λ is the number of conjugacy classes of subgroups $L \cong Q_8$ in G for which $3 \nmid |A_G(L)|$. (In case $n = 3$, replace λ in the expression for $P_G(t)$ by 2λ.)

Proof. Let us write $n^r_{H,V}$ for the multiplicity of $\mathcal{S}_{H,V}$ as a composition factor in M^r. On summing over r we obtain immediately

$$P_G(t) = \sum_{H,V} \Big(\dim \mathcal{S}_{H,V}(G) \cdot \sum_{r=0}^{\infty} t^r n^r_{H,V} \Big).$$

It is convenient to write

$$N_{H,V}(t) = \sum_{r=0}^{\infty} t^r n^r_{H,V}.$$

(a) Suppose G has D_{2^n} as a Sylow 2-subgroup. From our earlier discussion, the only pairs (H, V) which contribute to the sum are $(C_2 \times C_2, 2)$ and $(D_{2^n}, 1)$, and so

$$P_G(t) = \dim \mathcal{S}_{C_2\times C_2,2}(G) N_{C_2\times C_2,2} + \dim \mathcal{S}_{D_{2^n},1}(G) N_{D_{2^n},1}.$$

From equation 4.2 and the fact that $H^*(C_2 \times C_2, \mathbb{F}_2)$ is the polynomial algebra on the 2-dimensional simple $\mathbb{F}_2 S_3$-module, we obtain that $N_{C_2\times C_2,2}(t) = \sum_{r=0}^{\infty} t^r n^r_{C_2\times C_2,2}$ is the series in which $n^r_{C_2\times C_2,2}$ is the multiplicity of 2 as a composition factor of the rth symmetic power of 2. This series is calculated to be

$$N_{C_2\times C_2,2}(t) = \frac{t}{(1-t)(1-t^3)}$$

using a slight extension of Molien's theorem (see [3]) and the theory of Brauer characters. Taking the particular case of $G = D_{2^n}$ we have

$$\begin{aligned} P_{D_{2^n}}(t) &= \frac{1}{(1-t)^2} \\ &= \dim \mathcal{S}_{C_2\times C_2,2}(D_{2^n}) N_{C_2\times C_2,2}(t) + \dim \mathcal{S}_{D_{2^n},1}(D_{2^n}) N_{D_{2^n},1}(t) \\ &= 2N_{C_2\times C_2,2}(t) + N_{D_{2^n},1}(t). \end{aligned}$$

We deduce that

$$N_{D_{2^n},1}(t) = \frac{1+t^3}{(1-t^2)(1-t^3)}.$$

For a general group G with D_{2^n} as Sylow 2-subgroup we have

$$\mathcal{S}_{C_2\times C_2,2}(G) = \bigoplus_{C_2\times C_2 \cong L \le G} \mathrm{tr}_L^{N_G(L)}\, 2.$$

There are at most two conjugacy classes of $C_2 \times C_2$ subgroups in G, since D_{2^n} has two such classes; their normalizers in D_{2^n} are both D_8. Thus $N_G(L)/C_G(L)$ is either C_2 or S_3. Since $C_G(L)/L$ has odd order and acts trivially on the 2-dimensional module, $\mathrm{tr}_L^{C_G(L)}\, 2 = 2$. From the well-known structure of 2 as an $\mathbb{F}_2 S_3$-module we have $\dim \mathrm{tr}_L^{N_G(L)}\, 2 = 1$ or 0 according as $N_G(L)/C_G(L) \cong C_2$ or S_3. We deduce that $\dim \mathcal{S}_{C_2\times C_2,2}(G)$ equals the number of conjugacy classes of $C_2 \times C_2$ whose automizer has order not divisible by 3.

We compute also that

$$\mathcal{S}_{D_{2^n},1}(G) = \mathrm{tr}_{D_{2^n}}^{N_G(D_{2^n})}\, 1$$

has dimension 1 always, since $N_G(D_{2^n})/D_{2^n}$ has odd order. By putting these pieces together we obtain the statement of part (a) of the theorem.

(b) Suppose now that G has SD_{2^n} as a Sylow 2-subgroup. In this case

$$P_G(t) = \dim \mathcal{S}_{C_2\times C_2,2}(G) N_{C_2\times C_2,2} + \dim \mathcal{S}_{Q_8,2}(G) N_{Q_8,2} + \dim \mathcal{S}_{SD_{2^n},1}(G) N_{SD_{2^n},1}.$$

To compute $N_{Q_8,2}$ we use the equation

$$M^r(Q_8) = \mathcal{S}_{Q_8,1}(Q_8) n^r_{Q_8,1} + \mathcal{S}_{Q_8,2}(Q_8) n^r_{Q_8,2}$$

which holds in $G_0(\mathbb{F}_2[\mathrm{Out}(Q_8)])$, and the structure of $H^*(Q_8, \mathbb{F}_2)$ tells us that multiplicities $n^r_{Q_8,2}$ form the sequence $0, 1, 1, 0, 0, 1, 1, 0, \ldots$. Thus

$$N_{Q_8,2}(t) = \frac{t(1+t)}{1-t^4}.$$

Since we have already determined $N_{C_2\times C_2,2}$ we may obtain $N_{SD_{2^n},1}$ from any Poincaré series $P_G(t)$ where G has SD_{2^n} as a Sylow 2-subgroup, and we suggest either SD_{2^n} itself using the calculation in [5], or $GL(3,2)$ using Quillen's calculation [11]. Using SD_{2^n} we obtain

$$\begin{aligned} N_{SD_{2^n},1} &= P_{SD_{2^n}}(t) - N_{Q_8,2} - N_{C_2\times C_2,2} \\ &= \frac{1+t^5}{(1-t^3)(1-t^4)} \end{aligned}$$

using the information on the values of simple functors on G given in the matrix Ψ. Finally we obtain the Poincaré series for an arbitrary group with SD_{2^n} as a Sylow 2-subgroup by evaluation the simple global Mackey functors. We find that $\dim \mathcal{S}_{C_2\times C_2,2}(G) = \mu$ and $\dim \mathcal{S}_{Q_8,2}(G) = \lambda$, the numbers given in the statement of the theorem.

(c) We suppose that G has Q_{2^n} as a Sylow 2-subgroup, in which case

$$P_G(t) = \dim \mathcal{S}_{Q_8,2}(G) N_{Q_8,2} + \dim \mathcal{S}_{Q_2^n,1}(G) N_{Q_2^n,1}.$$

Taking $G = Q_{2^n}$ and noting that $\dim \mathcal{S}_{Q_8,2}(Q_2^n) = 2$, our existing knowledge of $N_{Q_8,2}$ gives

$$\begin{aligned} N_{Q_2^n,1} &= P_{Q_2^n}(t) - 2N_{Q_8,2} \\ &= \frac{(1+t+t^2)(1+t)}{1-t^4} - 2\frac{t(1+t)}{1-t^4} \\ &= \frac{1+t^3}{1-t^4}. \end{aligned}$$

This gives the desired result, noting that for general G with Q_{2^n} as Sylow 2-subgroup

$$\dim \mathcal{S}_{Q_2^n,1}(G) = 1 \quad \text{and} \quad \dim \mathcal{S}_{Q_8,2}(G) = \begin{cases} \lambda & \text{if } n \geq 4, \\ 2\lambda & \text{if } n = 3. \end{cases}$$

□

It is amusing to consider a power series whose evaluation at each group G is the Poincaré series $P_G(t)$ for that group. This series is

$$P(t) = \sum_{r=0}^{\infty} t^r M^r,$$

a power series in t with coefficients in Π, and it is a way of recording the composition factors of M^r, for every r. Since $M^r = \sum_{H,V} n^r_{H,V}\mathcal{S}_{H,V}$ we evidently have $P(t) = \sum_{H,V} N_{H,V}(t)\mathcal{S}_{H,V}$. The coefficient of t^r is a list of the composition factors of M^r, and on evaluation at each group G the coefficient of t^r is $H^r(G,\mathbb{F}_2)$, as an element of Π'. Summing up our calculations so far, together with some further ones, we have:

(4.4) THEOREM.

$$\begin{aligned}P(t) =&\frac{1}{1-t}\mathcal{S}_{C_2,1} + \frac{1}{1-t}\mathcal{S}_{C_4,1} + \frac{1+t^3}{(1-t^2)(1-t^3)}\mathcal{S}_{C_2\times C_2,1} + \frac{t}{(1-t)(1-t^3)}\mathcal{S}_{C_2\times C_2,2}\\
&+ \frac{1}{1-t}\mathcal{S}_{C_8,1} + \frac{1+t^3}{(1-t^2)(1-t^3)}\mathcal{S}_{D_8,1} + \frac{1+t^3}{1-t^4}\mathcal{S}_{Q_8,1} + \frac{t(1+t)}{1-t^4}\mathcal{S}_{Q_8,2}\\
&+ \frac{1-t+t^4-t^7+t^8}{(1-t)(1-t^3)(1-t^7)}\mathcal{S}_{C_2\times C_2\times C_2,1} + \frac{t(1-t^3+t^4+t^5)}{(1-t)(1-t^3)(1-t^7)}\mathcal{S}_{C_2\times C_2\times C_2,3a}\\
&+ \frac{t^2(1+t-t^2+t^5)}{(1-t)(1-t^3)(1-t^7)}\mathcal{S}_{C_2\times C_2\times C_2,3b} + \frac{t^4}{(1-t)(1-t^3)(1-t^7)}\mathcal{S}_{C_2\times C_2\times C_2,8}\\
&+ \frac{1+t^5}{(1-t^3)(1-t^4)}\mathcal{S}_{SD_{16},1} + \frac{1+t^2+t^3-t^4+t^5+t^6+t^8}{(1-t)(1-t^3)(1-t^7)}\mathcal{S}_{D_8\times C_2,8} + \cdots\end{aligned}$$

Remark. As well as having a use in the explicit computation of cohomology, the method just described may also be used to derive certain theoretical results such as Swan's theorem [12] on the cohomology of p-normal groups. We omit the details.

5. Projective covers of simple functors

Throughout this section we will assume that our coefficient ring R is a field or a complete discrete valuation ring, and in this situation our main results are that the simple functors have projective covers, and that the representable functors are finite direct sums of these. The theory works both for global Mackey functors and inflation functors. When we use it in the next section we will only need the case of inflation functors, and so we restrict attention to these functors now. However, the theory for global Mackey functors is just the same and may be obtained by a minor modification of the ideas, at times achieved by merely writing Ω_R instead of Ω_R^+, etc.

Since inflation functors are contravariant R-additive functors $\Omega_R^+ \to R$-mod, it is an immediate consequence of Yoneda's lemma that the representable functors $A_R(\ \ ,G) = \mathrm{Hom}_{\Omega_R^+}(\ \ ,G)$ are projective. We state Yoneda's lemma and its immediate well-known consequences in this context.

(5.1) PROPOSITION. *For each finite group G the representable functor $A_R(\ \ ,G)$ has the following properties.*

(i) $\mathrm{Hom}_{\mathrm{Mack}_R^*}(A_R(\ \ ,G), M) \cong M(G)$ *for each inflation functor M.*

(ii) In particular $\mathrm{Hom}_{\mathrm{Mack}_R^*}(A_R(\ \ ,G), A_R(\ \ ,H)) \cong A_R(G,H)$ *is a finitely generated R-module and* $\mathrm{End}_{\mathrm{Mack}_R^*}(A_R(\ \ ,G)) \cong A_R(G,G)$ *is an R-algebra finitely generated as an R-module.*

(iii) $A_R(\ \ ,G)$ *is a projective object in* Mack_R^*.

(iv) $A_R(\ \ ,G)$ *is generated as an inflation functor by its value at G.*

Proof. We prove only (iv). The argument of Yoneda's lemma is that if $\theta \in A_R(H,G)$ is any element then $\theta = A_R(\theta, G)(1_G)$ where $1_G : G \to G$ is the identity morphism in Ω_R^+. Since the subfunctor of $A_R(\ \ ,G)$ generated by 1_G must be closed under mappings $A_R(\theta, G)$ we deduce that θ lies in this subfunctor. □

We will also need the following preliminary lemma.

(5.2) LEMMA. $\mathrm{End}_{\mathrm{Mack}_R^*}(\mathbb{S}_{H,V}) \cong \mathrm{End}_{R[\mathrm{Out}(H)]}(V)$.

Proof. We have

$$\begin{aligned}\mathrm{End}_{\mathrm{Mack}^*}(\mathbb{S}_{H,V}) &\cong \mathrm{Hom}(\mathbb{S}_{H,V}, J_{H,V})\\ &\cong \mathrm{Hom}(\overline{\mathbb{S}_{H,V}}(H), V)\\ &= \mathrm{End}_{R[\mathrm{Out}(H)]}(V),\end{aligned}$$

the first isomorphism arising from the fact that $\mathbb{S}_{H,V}$ is the unique simple subfunctor of $J_{H,V}$ and the second from the adjoint property. □

We define $\operatorname{Rad} A_R(\ \ ,G)$ to be the intersection of the maximal subfunctors of $A_R(\ \ ,G)$, or in other words the intersection of the kernels of the homomorphisms from $A_R(\ \ ,G)$ to simple functors.

(5.3) LEMMA. *Let R be a field or a discrete valuation ring. Then*

$$A_R(\ \ ,G)/\operatorname{Rad} A_R(\ \ ,G) \cong \bigoplus \mathbb{S}_{H,V}^{d_{H,V}}$$

where $d_{H,V} = \dim \mathbb{S}_{H,V}(G)/\dim \operatorname{End}(V)$, and in the case of a discrete valuation ring the dimensions are taken over the residue field. This functor is a finite direct sum of simple functors, and its endomorphism ring is a finite dimensional semisimple algebra.

Proof. The multiplicity with which $\mathbb{S}_{H,V}$ appears as a homomorphic image of $A_R(\ \ ,G)$ is

$$\dim \operatorname{Hom}(A_R(\ \ ,G),\mathbb{S}_{H,V})/\dim \operatorname{End}(\mathbb{S}_{H,V}) = \dim \mathbb{S}_{H,V}(G)/\dim \operatorname{End}(V).$$

The number of simples which can be homomorphic images of $A_R(\ \ ,G)$ is thus finite, since only $\mathbb{S}_{H,V}$ with H isomorphic to a subgroup of G can appear, each with finite multiplicity, and for each group H there are only finitely many isomorphism classes of simple $R[\operatorname{Out}(H)]$-modules. Thus $A_R(\ \ ,G)/\operatorname{Rad} A_R(\ \ ,G)$ is isomorphic to the direct sum of all these simples, taken with their multiplicities. □

(5.4) THEOREM. *Assume that the ground ring R is a field or a complete discrete valuation ring. Every simple object $\mathbb{S}_{H,V}$ in* Mack* *has a projective cover, which is a projective inflation functor $\mathbb{P}_{H,V}$ such that $\mathbb{P}_{H,V}/\operatorname{Rad}\mathbb{P}_{H,V} \cong \mathbb{S}_{H,V}$.*

Proof. Let $\mathbb{S}_{H,V}$ be simple. Since $\mathbb{S}_{H,V}(H) \neq 0$ it follows from 5.3 that $\mathbb{S}_{H,V}$ is a direct summand of

$$A_R(\ \ ,H)/\operatorname{Rad} A_R(\ \ ,H)$$

and has the form $e(A_R(\ \ ,H)/\operatorname{Rad} A_R(\ \ ,H))$ for some idempotent endomorphism e.

Since $A_R(\ \ ,H)$ is projective, every endomorphism of $A_R(\ \ ,H)/\operatorname{Rad} A_R(\ \ ,H)$ lifts to an endomorphism of $A_R(\ \ ,H)$. Thus

$$\operatorname{End}(A_R(\ \ ,H)) \to \operatorname{End}(A_R(\ \ ,H)/\operatorname{Rad} A_R(\ \ ,H))$$

is an epimorphism, and by 5.3 the target ring is semisimple. Now $\operatorname{End}(A_R(\ \ ,H))$ is an algebra finitely generated as an R-module and so idempotents lift from $\operatorname{End}(A_R(\ \ ,H)/\operatorname{Rad} A_R(\ \ ,H))$ to $\operatorname{End}(A_R(\ \ ,H))$, preserving primitivity (see [3, 6.8]). Thus there is an idempotent endomorphism $\hat{e}$ of $A_R(\ \ ,H)$ which induces e on $A_R(\ \ ,H)/\operatorname{Rad} A_R(\ \ ,H)$, and $\hat{e}$ is primitive. The summand $\hat{e}(A_R(\ \ ,H)) = \mathbb{P}_{H,V}$ is now an indecomposable projective object in Mack* with the property that $\mathbb{P}_{H,V}/\operatorname{Rad}\mathbb{P}_{H,V} = \mathbb{S}_{H,V}$, and so this is the projective cover of $\mathbb{S}_{H,V}$. □

At this stage we may conclude that $A_R(\ \ ,G)$ has a set of summands $\mathbb{P}_{H,V}$ corresponding to the decomposition of $A_R(\ \ ,G)/\operatorname{Rad} A_R(\ \ ,G)$ as a direct sum of simples. These are precisely the direct summands of $A_R(\ \ ,G)$ which have a simple object as a homomorphic image. There remains the possibility, however, that $A_R(\ \ ,G)$ may have some summands which have no simple homomorphic image, and this is a possibility which we have to exclude. The property of $A_R(\ \ ,G)$ which we use is that it is generated by its value at G.

(5.5) LEMMA. *Let $M \in \text{Mack}^*$ be such that there is a finite set of groups $\mathcal{Y}$ closed under taking subgroups so that*
(i) $M(Y)$ has finite composition length as an $R[\operatorname{Out}(Y)]$-module for all $Y \in \mathcal{Y}$,
(ii) $M = \langle M(Y) \mid Y \in \mathcal{Y}\rangle$.
Then M is an epimorphic image of a finite direct sum of indecomposable projectives $\mathbb{P}_{H,V}$.

Proof. We proceed by induction on $\sum_{Y\in\mathcal{Y}} \ell(M(Y))$ where ℓ denotes composition length as an $R[\operatorname{Out}(H)]$-module. When this number is 0 condition (ii) implies that $M = 0$, so this starts the induction. Supposing now that some $M(Y) \neq 0$, pick a minimal group H for M which lies in $\mathcal{Y}$. There is a non-zero homomorphism $M(H) = \overline{M}(H) \to V$ for some simple $R[\operatorname{Out}(H)]$-module V, hence a non-zero homomorphism $M \to J_{H,V}$. Thus $\mathbb{S}_{H,V}$ is a composition factor of M, and there is a non-zero morphism $\mathbb{P}_{H,V} \to M$. Write M_1 for the cokernel of this map. It is a functor also satisfying conditions (i) and (ii) but now the sum of the composition lengths for M_1 is smaller than it was for M and so by induction there is a finite direct sum Q of indecomposable projectives $\mathbb{P}_{H,V}$ and an epimorphism $Q \to M_1$. Lift this map to a morphism $Q \to M$ using projectivity, and form $\mathbb{P}_{H,V} \oplus Q \to V$. This is necessarily an epimorphism. □

Remark. Any epimorphic image of a finite direct sum of $\mathbb{P}_{H,V}$'s also satisfies conditions (i) and (ii), so we obtain a characterisation of such epimorphic images as the functors satisfying (i) and (ii).

(5.6) THEOREM. *Let R be a field or a complete discrete valuation ring. Then $A_R(\ \ ,G)$ is the direct sum in* Mack* *of indecomposable projectives $\mathbb{P}_{H,V}$ where $\mathbb{P}_{H,V}$ occurs with multiplicity $\dim \mathbb{S}_{H,V}(G)/\dim \operatorname{End}(V)$, and where V is a simple $R[\operatorname{Out}(H)]$-module.*

Proof. From the argument in 5.4 we see that $\mathbb{P}_{H,V}$ does occur as a summand with the stated multiplicity, and it remains to show that no other summands occur. Such a summand, also by the argument of 5.4, would be a projective which has no simple functor as a homomorphic image. In case R is a discrete valuation ring, on dividing everything by the maximal ideal of R we would reduce to the same situation over the residue field of R, since by Nakayama's lemma the reduction of a non-zero module is non-zero. Thus to show the non-existence of such summands we may assume that R is a field. Now $A_R(\ \ ,G)$

satisfies conditions (i) and (ii) of 5.5 where we take $\mathcal{Y}$ to consist of all subgroups of G. This is because for each group Y, $A_R(Y,G)$ is finite dimensional from its definition, whence condition (i), and condition (ii) is satisfied because of 5.1(iv). It is immediate that every direct summand of $A_R(\ \ ,G)$ therefore also satisfies (i) and (ii) of 5.5 and so must be an epimorphic image of a direct sum of projectives $\mathbb{P}_{H,V}$. Such an epimorphic image would indeed have a simple functor as an epimorphic image. We deduce that there can be no remaining summand. □

6. The stable decomposition of BG

We now have a description of the indecomposable summands of $A_R(\ \ ,G)$ in the case of a field or a discrete valuation ring, and since this functor has endomorphism ring $A_R(G,G)$ these summands are in bijection with the primitive idempotents of $A_R(G,G)$. In case $G=P$ is a p-group and $R=\mathbb{Z}_p$ (or $\mathbb{F}_p$) it is a consequence of Carlsson's theorem pointed out by Lewis May and McClure [6] (see also [10]) that these idempotents are in bijection with the indecomposable stable wedge summands of $(BP_+)^\wedge_p$. By BP_+ we will mean the suspension spectrum of the classifying space BP after a disjoint base point has been adjoined, and we will only ever consider p-completions $(BP_+)^\wedge_p$. For a p-group P, BP_+ is in fact already p-complete except for a sphere spectrum summand. We regard these classifying space spectra as objects in the category $\mathcal{S}$ of all p-completed suspension spectra with homotopy classes of stable maps. As stated in [10, Cor. 15] , when P is a p-group the group of homomorphisms $\mathrm{Hom}_{\mathcal{S}}((BP_+)^\wedge_p,(BG_+)^\wedge_{\mathrm{i}}) = [BP_+,BG_+]^\wedge_p$ is isomorphic to $A_{\mathbb{Z}_p}(P,G)$, and when $G=P$ we have a ring isomorphism. From this one not only sees that primitive idempotents of $A_{\mathbb{Z}_p}(P,P)$ are in bijection with the indecomposable summands of $(BP_+)^\wedge_p$, but also that two summands are isomorphic in $\mathcal{S}$ precisely if the idempotents are conjugate, which in turn happens precisely if the corresponding indecomposable projective summands of $A_{\mathbb{Z}_p}(\ \ ,P)$ are isomorphic. We wish to say more than this, namely to say when two summands of $(BG_+)^\wedge_p$ and $(BH_+)^\wedge_p$ are isomorphic for different groups G and H, thereby making the connection with the theory of dominant summands discussed in [1]. One easily guesses the answer for p-groups: two such stable summands of $(BG_+)^\wedge_p$ and $(BH_+)^\wedge_p$ are isomorphic precisely if the corresponding indecomposable projective summands of $A_{\mathbb{Z}_p}(\ \ ,G)$ and $A_{\mathbb{Z}_p}(\ \ ,H)$ are isomorphic. The proof of this relies on the following straightforward observation.

(6.1) PROPOSITION. *Let p be a prime.*

(i) The full subcategory $\mathcal{A}$ of $\mathcal{S}$ whose objects are the classifying spaces $(BP_+)^\wedge_p$ of finite p-groups P is equivalent to the full subcategory $\mathcal{B}$ of $\mathrm{Mack}^*_{\mathbb{Z}_p}$ whose objects are the representable functors $A_{\mathbb{Z}_p}(\ \ ,P)$, where again P ranges over finite p-groups.

(ii) *The full subcategory $\mathcal{C}$ of $\mathcal{S}$ whose objects are stable summands of the classifying spaces $(BP_+)^\wedge_p$ with P a p-group is equivalent to the full subcategory $\mathcal{D}$ of* $\mathrm{Mack}^*_{\mathbb{Z}_p}$ *whose objects are the indecompsable projectives $\mathbb{P}_{H,V}$ with H a p-group.*

When G is an arbitrary finite group it is well-known (see e.g. [1]) that $(BG_+)^\wedge_p$ is isomorphic to a summand of $(BP_+)^\wedge_p$ where P is a Sylow p-subgroup of G. Thus the category $\mathcal{C}$ just defined in 6.1(ii) equals the full subcategory of $\mathcal{S}$ whose objects are the stable summands of all $(BG_+)^\wedge_p$, ranging over all finite groups G.

Proof. (i) We define a functor $\mathcal{A} \to \mathcal{B}$ by $(BP_+)^\wedge_p \mapsto A_{\mathbb{Z}_p}(\quad, P)$ on objects, and on morphisms by means of the isomorphism

$$\mathrm{Hom}_{\mathcal{S}}((BP_+)^\wedge_p, (BQ_+)^\wedge_p) \cong A_{\mathbb{Z}_p}(P, Q) \cong \mathrm{Hom}_{\mathrm{Mack}^*}(A_{\mathbb{Z}_p}(\quad, P), A_{\mathbb{Z}_p}(\quad, Q)).$$

There is an inverse functor given on objects simply by $A_{\mathbb{Z}_p}(\quad, P) \mapsto (BP_+)^\wedge_p$. One has to check that the isomorphism between the Hom sets is functorial, and indeed composition of morphisms in $\mathcal{A}$ and in $\mathcal{B}$ does correspond to the operation

$$A_{\mathbb{Z}_p}(P, Q) \times A_{\mathbb{Z}_p}(Q, K) \to A_{\mathbb{Z}_p}(P, K)$$

described in [10] or [1].

(ii) One may construct the category $\mathcal{C}$ from $\mathcal{A}$ as follows. The objects of $\mathcal{C}$ may be identified with the idempotents $e \in \mathrm{End}_{\mathcal{S}}((BP_+)^\wedge_p)$, as P varies. Now if $f \in \mathrm{End}_{\mathcal{S}}((BQ_+)^\wedge_p)$ is another idempotent then $\mathrm{Hom}_{\mathcal{C}}(e, f) = f \cdot A_{\mathbb{Z}_p}(P, Q) \cdot e$.

By an exactly similar construction we may obtain $\mathcal{D}$ from $\mathcal{B}$. Since $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, we deduce that $\mathcal{C}$ and $\mathcal{D}$ are isomorphic. □

(6.2) THEOREM. *For any finite group G the indecomposable stable summands of $(BG_+)^\wedge_p$ are in bijection with those summands $\mathbb{P}_{H,V}$ of $A_{\mathbb{Z}_p}(\quad, G)$ for which H is a p-group. Here V is a simple $\mathbb{F}_p[\mathrm{Out}(H)]$-module, and the multiplicity of a summand parametrized by (H, V) as a summand of $(BG_+)^\wedge_p$ equals the multiplicity of $\mathbb{P}_{H,V}$ as a summand of $A_{\mathbb{Z}_p}(\quad, G)$, which equals $\dim_{\mathbb{F}_p} \mathbb{S}_{H,V}(G) \,/\, \dim_{\mathbb{F}_p} \mathrm{End}(V)$. This number may be computed using 5.5 and 5.6. In particular, for each indecomposable summand X of a p-completed classifying space $(BG_+)^\wedge_p$ there is a unique isomorphism class of groups H minimal such that X is a summand of $(BH_+)^\wedge_p$, and H is a p-group. Such a summand is parametrized by a pair (H, V) and its multiplicity as a summand of $(BH_+)^\wedge_p$ is $\dim V / \dim \mathrm{End}_{\mathbb{F}_p \mathrm{Out}(H)}(V)$.*

Proof. To deal with the first part of the statement, let P be a Sylow p-subgroup of G. As indicated in [1], the maps $Bi : (BP_+)^\wedge_p \to (BG_+)^\wedge_p$ and the transfer map $(BG_+)^\wedge_p \to (BP_+)^\wedge_p$ are equivalences between $(BG_+)^\wedge_p$ and a summand of $(BP_+)^\wedge_p$. Thus the composite of these maps is an endomorphism of $(BP_+)^\wedge_p$ which has as its image a summand isomorphic to $(BG_+)^\wedge_p$. In the category of functors this corresponds to the composite

$$A_{\mathbb{Z}_p}(\quad, P) \xrightarrow{{}_PG_G} A_{\mathbb{Z}_p}(\quad, G) \xrightarrow{{}_GG_P} A_{\mathbb{Z}_p}(\quad, P)$$

as is also explained in [1], although here the middle term $A_{\mathbb{Z}_p}(\quad, G)$ no longer corresponds to $(BG_+)^\wedge_p$. The map denoted ${}_PG_G$ is surjective on evaluation at all p-subgroups Q of G since in the interpretation of $A_{\mathbb{Z}_p}(Q, G)$ as being the free $\mathbb{Z}_p$-module with basis the equivalence classes of pairs (K, ϕ) where $K \leq Q$ and $\phi : K \to G$ (see [6], [10] or [1]), it is clear that ϕ can always be factored $\phi : K \to P \to G$, so that (K, ϕ) is the image of an element of $A_{\mathbb{Z}_p}(P, G)$. Similarly this factorisation implies that ${}_GG_P$ is injective on $A_{\mathbb{Z}_p}(Q, G)$ since equivalence of a pair (K, ϕ) as a map to G implies equivalence as a map to P.

The above argument shows that all summands of $A_{\mathbb{Z}_p}(\quad, G)$ of the form $\mathbb{P}_{H,V}$ with H a p-group are in the image of ${}_PG_G$, since these summands are generated by their values at p-groups. On the other hand the image of this map is isomorphic to a summand of $A_{\mathbb{Z}_p}(\quad, P)$, and so can be a direct sum only of such $\mathbb{P}_{H,V}$.

The remaining parts of the theorem now follow from 5.6 and 2.1. □

This theorem together with the information about the simple inflation functors given in Section 2 encapsulates many of the principal results of [1] and [8]. The theory of 'dominant summands' of [1] translates into the statement that the classifying space summand parametrized by the pair (H, V) is not a summand of $(BG_+)^\wedge_p$ unless H is isomorphic to a subgroup of G, and this summand does appear with non-zero multiplicity as a summand of BH_+, as stated in 6.2. From 2.7(ii) we obtain the criterion of [1] that the summand parametrized by (H, V) appears with non-zero multiplicity in $(BG_+)^\wedge_p$ if and only if one of the expressions $\sum \theta_g^{-1}$ considered there acts in a non-zero fashion on V. From 2.8(ii) we obtain a necessary condition for the summand parametrized by (H, V) to appear as a summand of $(BG_+)^\wedge_p$

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