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Weight Theory in the Context of Arbitrary Finite Groups

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Abstract. Many constructions which arise in the study of Mackey functors bear a similarity to certain constructions in Lie theory. We survey these similarities and indicate applications to the computation of group cohomology and the stable decomposition of classifying spaces, as well the connection with Alperin's weight conjecture.

1 Introduction

The objects of Lie theory give rise to some rather special algebraic and combinatorial structures which at first sight appear to be limited in their applicability to their Lie-theoretic origins. We have in mind, for example, the combinatorics of Weyl groups, Coxeter complexes and buildings, and the weight-theoretic description of representations. The theme of this survey is that many of these notions do have much wider applicability, and our particular goal is to outline some aspects of weight theory as it can be applied to representations of arbitrary finite groups (not just the finite groups of Lie type). We will confine ourselves to this, and will not discuss the ways in which other Lie-theoretic notions may be extended. For instance, some aspects of the theory of buildings have been made to work in broad circumstances, and for a (rather old) description of some of this see [19].

Before describing the way in which weight theory appears with finite groups, let us first take a moment to describe the features we might hope to find in such a theory. These features are expressed in their purest form with the representations of finite-dimensional semisimple complex Lie algebras, or more generally, of Kac-Moody Lie algebras. Here it is convenient to work with the BGG category $\mathcal{O}$ (see [12] or [16] for background). The simple (or irreducible) modules in this category are parametrized by a set of 'weights', which form a partially-ordered set. The simple

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modules may be constructed as the unique simple quotients of Verma modules, which are also parametrized by the weights, and these in turn may be constructed by starting with a 1-dimensional representation of the Cartan subalgebra given by a weight, inflating it to a Borel subalgebra and then inducing it to the whole Lie algebra. With respect to this parametrization of the simples $\mathcal{O}$ is a highest weight category [7], and this structure is given in a particular way: each simple module is a direct sum of weight spaces for the Cartan subalgebra (as part of the definition of $\mathcal{O}$), on each of which the Cartan subalgebra acts via a weight, and the highest of these weights is the one which parametrizes the simple module. The dimensions of the weight spaces of the Verma modules are easy to calculate, but the corresponding dimensions of the weight spaces of the simple modules are not. This information for the simple modules and the Verma modules is related via the Weyl character formula and also via the Kazhdan-Lusztig conjecture (now a theorem), which gives a formula for composition factor multiplicities of the Verma modules.

We will present a framework to do with representations of arbitrary finite groups in which some of this structure can be duplicated, and in which useful questions can be asked. This is the framework of Mackey functors. In studying Mackey functors some years ago we have observed that the simple Mackey functors may be parametrized by a certain set [17]. Since that time, further aspects of this parametrization have become apparent, which are akin to Lie theory. With hindsight this can be seen to be no accident, because Mackey functors are representations of a set of operators which satisfy relations similar in some ways to those satisfied by a Kac-Moody Lie algebra.

The parametrization of the simple Mackey functors (to be described later) bears an obvious resemblance to the weights which appear in Alperin's weight conjecture (see [1] and [2]). We have been very much motivated by Alperin's conjecture and state it now. It applies to the representations of a finite group G over an algebraically closed field k of characteristic $p > 0$. Alperin defines a *weight* to be a pair (H, V) where H is a p-subgroup of G and V is a simple projective $k[N_G(H)/H]$-module. Weights are taken up to isomorphism of the module V and conjugacy of the subgroup H. Alperin's conjecture is that the number of weights for G equals the number of (isomorphism types of) simple kG-modules. It is generally felt that this remarkable conjecture is true, and is the manifestation of something quite profound in representation theory, perhaps suggesting structure that has not yet been identified. It has certainly motivated a tremendous amount of work in modular representation theory and provided developments which are of interest in themselves, even though the conjecture has not yet been proven.

Alperin was careful to point out that in general we should not expect a canonical bijection between the weights and the simple kG-modules, so we should not think of these weights as parametrizing the simples. Why, then, should this suggest Lie theory? One reason is that when G is a group of Lie type in defining characteristic p, it is indeed possible to give a bijection between Alperin's weights and the weights (for example, in the sense of Curtis and Richen, see [9]) which usually parametrize the simple modules. It is the case that Alperin's weights are a subset of the set which parametrizes the simple Mackey functors. The fact that this parametrization of the larger set of simple Mackey functors has aspects of a weight theory reinforces the suggestion of Lie theory.

We have also been motivated in our study of Mackey functors by the stable decomposition of classifying spaces. Here we work with p-complete spectra for some fixed prime p, and let BG denote the p-completed suspension spectrum obtained from the classifying space of G. In this situation, relying on Carlsson's theorem (the Segal conjecture), the complete decomposition of BG as a wedge of indecomposable summands has been worked out for each group G (see [4], [3], [13], [14], [20]), at least in a theoretical sense. It turns out that the possible indecomposable summands which can occur are in bijection with pairs (H, V) where H is a p-group, and V is a simple $\mathbb{F}_p[\operatorname{Out} G]$-module, where $\operatorname{Out}(G) = \operatorname{Aut}(G)/\operatorname{Inn}(G)$ is the outer automorphism group of G. The spectrum parametrized by (H, V) does occur as a summand of BH, and if this spectrum occurs as a summand of some BG, then necessarily H occurs as a subgroup of G. Thus H is the unique minimal group among the groups G for which this spectrum occurs as a summand of BG. We will see that exactly the same condition arises when we consider the structure of the simple Mackey functors, and it expresses the fact that a simple Mackey functor has a unique highest weight (in a sense to be defined). It was these observations that motivated the approach in [20] where the problem of analyzing the indecomposable summands of BG is translated into the language of Mackey functors, and a proof of the main theorems of [4] and [14] is given using this theory.

Let us now briefly summarize the essential features of Mackey functors, which we will explain more fully in the later sections. There are two kinds of Mackey functor, one kind defined only on the subgroups of a fixed group G (which we will call here *ordinary Mackey functors*), and the second kind defined on all finite groups (called *globally-defined Mackey functors*). The theories go in parallel, and it is the first kind which is appropriate for Alperin's conjecture, the second for the stable decomposition of classifying spaces. In both cases the simple Mackey functors are parametrized by pairs (H, V) where, in the first case, H is a subgroup of G up to conjugacy and V is a simple $N_G(H)/H$-module, and in the second case H is a group taken up to isomorphism and V is a simple $\operatorname{Out} H$-module. Thus the sets of pairs which arise in Alperin's conjecture in the first case and in the stable decomposition of classifying spaces in the second case are subsets of the full sets of pairs which occur with Mackey functors.

We will write $W(H)$ for $N_G(H)/H$ in the case of ordinary Mackey functors, and for $\operatorname{Out} H$ in the case of globally-defined Mackey functors. Each Mackey functor may be viewed as a module for a certain algebra (the *Mackey algebra*), this module being the direct sum of all the evaluations $M(H)$ over various groups H. These $M(H)$ are what should be regarded as analogues of weight spaces in Lie theory. Each $M(H)$ is a module for $W(H)$, and there is a difference here with representations of a Lie algebra in that each of these 'weight spaces' $M(H)$ is a representation of a different group $W(H)$, and furthermore these representations need not be completely reducible. We will regard a pair (H, V), where V is a representation of $W(H)$, as a weight. Given a weight we may construct a Mackey functor $\Delta_{H,V}$ analogously to the construction of a Verma module. The adjoint properties of this construction mean that if V is a module with a unique simple quotient then $\Delta_{H,V}$ has a unique simple quotient as a Mackey functor, and by extracting the simple quotient $S_{H,V}$ when V is simple, this process provides a construction of all the simple Mackey functors in a way anologous to the construction of the simple modules with highest weight for a Kac-Moody Lie algebra. In the Mackey functor situation, the simple functor $S_{H,V}$ parametrized by (H, V) has evaluations which are only non-zero on groups which are bigger than H. We will put a partial order on weights by requiring that $(H, V) < (J, W)$ if and only if J is a homomorphic image of a subgroup of H and is smaller than H, and now we may say that $S_{H,V}$ has a unique highest weight space, with weight (H,V).

The parallel with Lie theory prompts us to ask, for example, whether Mackey functors form a highest weight category. The answer is that they do not in general, but we point out situations in which they do. However, it turns out that certain of the $\Delta_{H,V}$ always satisfy a vanishing condition for their Ext groups analogous to a crucial property satisfied by the Δ modules in a highest weight category, and this allows us to proceed with a construction, due to Ringel in the case of highest weight categories. We work with the full subcategory of Mackey functors in which the objects are the Mackey functors which have a filtration with $\Delta_{H,V}$ factors where V is a p-permutation module. In this context we are able to imitate the construction of the 'Ringel dual', and the algebra we obtain is standardly stratified in the sense of [8].

In the remaining sections we explain this theory in greater detail, concluding with some comments on Alperin's weight conjecture.

2 Definitions and Examples of Mackey Functors

A more complete introduction to Mackey functors can be found in [21] where many references are given. In the next sections we will describe constructions which are explained in [17], [18], [5] and [22]. Here we start with the definitions.

These days there are two kinds of Mackey functor: those which take values on the subgroups of a single group, and those which are defined on a bigger class of groups, usually all finite groups. The first (original) kind derives from definitions of Dress [10] and Green [11]. Working over a commutative ring R with a 1 we define an (ordinary) *Mackey functor* on a finite group G to be made up of the following specification:

- For each subgroup H of G there is an R-module $M(H)$.
- For each inclusion of subgroups $\alpha : H \hookrightarrow K \leq G$ there are R-module homomorphisms $\alpha_* : M(H) \to M(K)$ and $\alpha^* : M(K) \to M(H)$.
- For each $g \in G$ and subgroup $H \leq G$ there is an R-module homomorphism $c_g : M(H) \to M(gHg^{-1})$.
- These operations are functorial, in that $(\alpha\beta)_* = \alpha_*\beta_*$, $(\alpha\beta)^* = \beta^*\alpha^*$, $c_g c_h = c_{gh}$, and also $\alpha_* c_g = c_g \alpha_*$, $\alpha^* c_g = c_g \alpha^*$ whenever these composites are defined.
- $c_{g^{-1}} = c_g^{-1}$ always.
- If $g \in H$ then $c_g : M(H) \to M(gHg^{-1})$ is the identity homomorphism.
- The Mackey formula holds: writing ι_H^K for the inclusion $H \hookrightarrow K$, whenever H and K are subgroups of $J \leq G$ we always have

  $$(\iota_K^J)^*(\iota_H^J)_* = \sum_{g\in[K\backslash J/H]} (\iota_{K\cap {}^gH}^K)_* c_g (\iota_{K^g\cap H}^H)^*.$$

  In this formula we write ${}^gH = gHg^{-1}$, $H^g = g^{-1}Hg$ and $[K\backslash J/H]$ denotes a set of representatives for the (K,H)-double cosets in J.

Some standard examples of Mackey functors M on G are $M(J) = R(J)$, the ring of ordinary characters of J, and $M(J) = H^n(J,V)$, the degree n cohomology of J with coefficients in an RG-module V, for some fixed n. In these examples, α_* is induction, α^* is restriction and c_g is conjugation by g.

Globally-defined Mackey functors have a similar definition, but we now allow them to be defined on all finite groups (or sometimes a subclass of finite groups) and we allow operations α_* and α^* defined on a broader class of homomorphisms of these groups. Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of groups closed under taking quotients of subgroups. For example, they could consist of all finite groups, or just the identity group, or all nilpotent groups or one of many other possibilities. A *globally-defined Mackey functor* is given by the following specification:

- For each group G there is an R-module $M(G)$.
- For each group homomorphism $\alpha : H \to K$ with $\ker\alpha \in \mathcal{Y}$ there is an R-module homomorphism $\alpha_* : M(H) \to M(K)$.
- For each group homomorphism $\alpha : H \to K$ with $\ker\alpha \in \mathcal{X}$ there is an R-module homomorphism $\alpha^* : M(K) \to M(H)$.
- These operations are functorial, in that $(\alpha\beta)_* = \alpha_*\beta_*$, $(\alpha\beta)^* = \beta^*\alpha^*$ whenever these composites are defined.
- Whenever $\alpha : G \to G$ is an inner automorphism then $\alpha_* = 1 = \alpha^* : M(G) \to M(G)$.
- For every commutative diagram of groups

$$\begin{array}{ccc} G & \xrightarrow{\beta} & H \\ \gamma\uparrow & & \uparrow\alpha \\ \beta^{-1}(K) & \xrightarrow{\delta} & K \end{array}$$

  in which α and γ are inclusions and β and δ are surjections we have $\alpha^*\beta_* = \delta_*\gamma^*$ whenever $\ker\beta \in \mathcal{Y}$, and $\beta^*\alpha_* = \gamma_*\delta^*$ whenever $\ker\beta \in \mathcal{X}$.
- For every commutative diagram of groups

$$\begin{array}{ccc} G & \xrightarrow{\gamma} & H/\ker\alpha\ker\beta \\ \beta\uparrow & & \uparrow\delta \\ H & \xrightarrow{\alpha} & K \end{array}$$

  in which α, β, γ and δ are all surjections, with $\ker\beta \in \mathcal{Y}$ and $\ker\alpha \in \mathcal{X}$, we have $\beta_*\alpha^* = \gamma^*\delta_*$.
- The Mackey formula holds: whenever H and K are subgroups of J we always have

$$(\iota_K^J)^*(\iota_H^J)_* = \sum_{g\in[K\backslash J/H]} (\iota_{K\cap {}^gH}^K)_* c_{g*} (\iota_{K^g\cap H}^H)^*.$$

  where $c_g : K^g \cap H \to K \cap {}^gH$ is the homomorphism $c_g(x) = gxg^{-1}$ and $\iota_H^J : H \hookrightarrow J$ denotes inclusion.

Evidently there are similarities and differences between the definitions of the two types of Mackey functor. With a globally-defined Mackey functor the operations α_* and α^* are always defined when α is an injective group homomorphism (the only situation considered with ordinary Mackey functors), and they may also be defined on further group homomorphisms, depending on the choices of $\mathcal{X}$ and $\mathcal{Y}$. Because of this extra generality, the globally-defined Mackey functors have two extra axioms which describe the interaction between injective and surjective homomorphisms.

It is the case that every globally-defined Mackey functor, when restricted to the subgroups of a particular group G and the inclusion and conjugation homomorphisms, yields an ordinary Mackey functor. On the other hand, if G is any non-identity group, there are ordinary Mackey functors for G which are not the restriction of any globally-defined Mackey functor.

As examples of globally-defined Mackey functors we may take, as before, $M(G) = R(G)$, the character ring of ordinary characters of G. We may also take variations on this such as $M(G) = R_{\mathbb{Q}}(G)$, the ring of characters of $\mathbb{Q}G$-modules. Here we may allow $\mathcal{X} = \mathcal{Y} =$ all finite groups, imposing no restriction on the kind of operations α_* and α^* which we allow, for no matter what the homomorphism $\alpha : H \to K$ is, we can always restrict characters of K along the homomorphism to give characters of H; and given a representation of H we may tensor it over the group ring of H with the group ring of K to give a representation of K. It would also be possible to restrict the kind of operations we allow by taking $\mathcal{X}$ or $\mathcal{Y}$ to be proper classes of groups. We might do this if we had some application in mind where it is more appropriate not to consider the full range of operations available.

Another example of a globally-defined Mackey functor is the degree n cohomology $M(G) = H^n(G, R)$ for some fixed n, where R is the trivial module. In this example we must take $\mathcal{Y} = 1$, while $\mathcal{X}$ may be allowed to be all finite groups, because whereas cohomology has arbitrary contravariant operations (defined using the inflation map when a surjective homomorphism is involved), it only has covariant operations defined for injective group homomorphisms. Note that we take the trivial module here so that cohomology can be defined for all groups G. If we were to take a non-trivial G-module V then in general the ordinary Mackey functor $H^n(G, V)$ defined on the subgroups of G would not extend to a globally-defined Mackey functor.

For each group J the Mackey functor operations of the form $M(J) \to M(J)$ act as a group of automorphisms of $M(J)$, isomorphic to $N_G(J)$ in the case of ordinary Mackey functors, and to $\mathrm{Aut}(J)$ in the case of globally-defined Mackey functors. In the ordinary case it is an axiom that J itself acts trivially on $M(J)$, and in the globally-defined case it is an axiom that the inner automorphisms act trivially, so that $M(J)$ is an $R[N_G(J)/J]$-module when M is an ordinary Mackey functor, and an $R[\mathrm{Out}\, J]$-module when M is globally-defined. We will write

$$W(J) = \begin{cases} N_G(J)/J & \text{if } M \text{ is an ordinary Mackey functor,} \\ \mathrm{Out}\, J & \text{if } M \text{ is globally-defined,} \end{cases}$$

so that in either case $M(J)$ is an $R[W(J)]$-module.

We now define a *weight* in this context to be a pair (H, V) where H is a subgroup of G (in the case of ordinary Mackey functors), or just a group (in the case of globally-defined Mackey functors) and V is an $R[W(H)]$-module. We regard two weights (H, V), (K, U) as being the same if H and K are G-conjugate (ordinary Mackey functors) or isomorphic (globally-defined Mackey functors) and after identifying H and K via such an isomorphism V and U are isomorphic $R[W(H)]$-modules. We put a partial order on weights by requiring $(H, V) < (K, U)$ if and only if[1] K is a homomorphic image of a subgroup of H and $K \not\cong H$.

We see also that M is really a representation of a quiver with relations where the vertices of the quiver are the subgroups of G (ordinary Mackey functors) or a

[1] There is no condition on V and U in the definition of $<$. The definition we give works for both ordinary and globally-defined Mackey functors, but with ordinary Mackey functors it would be possible to use the more stringent definition $(H, V) < (K, U) \Leftrightarrow K$ is G-conjugate to a proper subgroup of H.

set of representatives of the isomorphism classes of finite groups (globally-defined Mackey functors), the arrows are the operations α_*, α^*, c_g which appear in the definition of M (using c_g only if M is an ordinary Mackey functor), and where the relations are the identities which appear in this definition. This is an approach explained in [18]. The corresponding path algebra modulo the ideal of relations is an algebra $\mu_R(G)$ of finite R-rank in the case of ordinary Mackey functors, and an algebra μ_R of infinite R-rank in the case of globally-defined Mackey functors. We will call these algebras *Mackey algebras.* The Mackey functor M can be regarded as a module for the Mackey algebra, and this module is the direct sum of all the evaluations $M(J)$. It is these evaluations which, we suggest, behave in the manner of weight spaces. We will see this when we come to construct the analogues of Verma modules, but for now we explain at a philosophical level why we might expect this.

A finite-dimensional complex semisimple Lie algebra may be generated by elements often denoted e_i, f_i, h_i, satisfying relations $[h_i, h_j] = 0$, $[e_i, f_i] = h_i$, $[e_i, f_j] = 0$ if $i \neq j$, $[h_i, e_j] = \alpha_j(h_i)e_j$, $[h_i, f_j] = -\alpha_j(h_i)f_j$ where α_j is the jth simple root, and together with two further relations expressing the nilpotency of $\operatorname{ad} e_i$ and $\operatorname{ad} y_i$ we obtain defining relations for the Lie algebra, according to a theorem of Serre. A straightforward calculation shows that if $V = \bigoplus V_\mu$ is a representation of the Lie algebra which is diagonalizable for the Cartan subalgebra, and thus a direct sum of weight spaces V_μ, then if $v \in V_\mu$ is a weight vector so are $h_i v \in V_\mu$, $e_i v \in V_{\mu+\alpha_i}$ and $f_i v \in V_{\mu-\alpha_i}$. We may express this situation schematically in the diagram

$$\begin{array}{ccccc} V_{\mu-\alpha_i} & \stackrel{f_i}{\longleftarrow} & V_\mu & \stackrel{e_i}{\longrightarrow} & V_{\mu+\alpha_i} \\ & & \circlearrowleft & & \\ & & h_i & & \end{array}$$

and we see that the operators which generate the Lie algebra preserve weight spaces, sending each weight space to another one so that the corresponding weights are comparable (in the usual partial order on weights).

With Mackey functors there is a similar diagram. Given group homomorphisms $H \stackrel{\alpha}{\longrightarrow} J \stackrel{\beta}{\longrightarrow} K$ and an automorphism $J \stackrel{\gamma}{\longrightarrow} J$ (which should be conjugation in the case of ordinary Mackey functors), for any Mackey functor M we have a diagram of Mackey functor operations

$$\begin{array}{ccccc} M(H) & \stackrel{\alpha^*}{\longleftarrow} & M(J) & \stackrel{\beta_*}{\longrightarrow} & M(K) \\ & & \circlearrowleft & & . \\ & & \gamma_* & & \end{array}$$

Not only are the 'weight spaces' preserved, but provided we consider only α and β which are either injective or surjective, these operations send weight spaces to weight spaces with weights which are comparable in the previously given partial order. Note that since every group homomorphism is a composite of a surjection and an injection, such operations α^* and β_* do generate the Mackey algebras $\mu_R(G)$ and μ_R. This interaction between the way the generators of the Mackey algebras operate and the partial order on weights is a key ingredient in allowing the construction of Mackey functors similar to Verma modules.

3 The Simple Mackey Functors and Δ Mackey Functors

Ordinary Mackey functors on a group G over a ring R form a category which we denote $\mathrm{Mack}_R(G)$ and which we may identify with the category of modules for the Mackey algebra $\mu_R(G)$. Similarly we denote the category of globally-defined Mackey functors by Mack_R and identify it with the category of μ_R-modules. These globally-defined functors depend on classes of groups $\mathcal{X}$ and $\mathcal{Y}$, but it is convenient to suppress these symbols from the notation. The categories of Mackey functors are abelian, and so we may talk about filtrations of Mackey functors, the factors in such a filtration, simple Mackey functors, and so on.

We present now some results about these objects, starting with the simple ones. The simple ordinary Mackey functors were parametrized in [17]. In the globally-defined case they were parametrized in a special situation in [20], and then generally in [5].

Theorem 3.1 (Bouc, Thévenaz, Webb) *Every simple Mackey functor S has a unique highest weight (H, V) such that $V = S(H)$ is non-zero. The module V appearing in the highest weight is a simple $R[W(H)]$-module, and writing $S_{H,V} = S$ we obtain a parametrization of the simple Mackey functors by the set of weights (H, V) where V is simple and H is arbitrary.*

The inital statement of this theorem may be rephrased by saying that each simple Mackey functor has a unique smallest group H on which it is non-zero, and its evaluation at H is a simple module. In the case of ordinary Mackey functors on a group G, H should of course be a subgroup of G, and it is unique up to conjugacy in G. In the case of globally-defined Mackey functors H may be any group, taken up to isomorphism.

Calculating the structure of the 'weight spaces' $S_{H,V}(J)$ is an important way of understanding what the simple Mackey functors look like. In the case of simple modules for a Lie algebra which lie in $\mathcal{O}$ this information is contained in the formal character of the module, which is really a list of the dimensions of the weight spaces. Because the representations of $W(J)$ need not be semisimple, the appropriate technique with Mackey functors is to consider $S_{H,V}(J)$ as an element of a Grothendieck group of $R[W(J)]$-modules (either the usual Grothendieck group of finitely-generated modules, or the Green ring) and to regard the 'formal character' of $S_{H,V}(J)$ as the list of such elements of Grothendieck groups. This approach was adopted in [18] and [20] where it was used in various calculations, including the development of a decomposition theory for Mackey functors [18], and a method for computing group cohomology in [20]. The multiplicites of stable summands of classifying spaces BG are also determined from this 'formal character', as explained in [20].

The explicit calculation of the modules $S_{H,V}(J)$ is generally not straightforward, although it can be done more easily in some cases than others. A (not entirely explicit) formula for $S_{H,V}(J)$ is given in [17] and [18] in the case of ordinary Mackey functors. Expressions for $S_{H,V}(J)$ in the case of globally-defined functors when at least one of the classes $\mathcal{X}$ and $\mathcal{Y}$ are the identity group are given in [20]. When $\mathcal{X}$ and $\mathcal{Y}$ are both the classes of all finite groups, the problem of determining the modules $S_{H,V}(J)$ seems to be very hard. As an example (due to Bouc and Thévenaz [5], [6]) let us take $\mathcal{X}$ and $\mathcal{Y}$ to be the classes of all finite groups and put $R = \mathbb{Q}$. The most accessible simple Mackey functor here is $S_{1,\mathbb{Q}}$, and Bouc shows that it is isomorphic to $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}$, where $R_{\mathbb{Q}}(G)$ is the ring of characters of $\mathbb{Q}G$-modules. This simple functor has a projective cover, and it is $\mathbb{Q} \otimes_{\mathbb{Z}} B$ where B is the Mackey functor which assigns to a group G its Burnside ring $B(G)$. Bouc conducts an analysis of the remaining simple Mackey functors which may appear as quotients of subfunctors of the projective Mackey functor $\mathbb{Q} \otimes_{\mathbb{Z}} B$ and shows that they are all of the form $S_{H,\mathbb{Q}}$ for certain groups H. If H happens to be a p-group for some prime p, the only possibility is that $H = C_p \times C_p$. Bouc and Thévenaz show in [6] that if G is a p-group then $S_{C_p \times C_p, \mathbb{Q}}(G)$ is Dade's group of endopermutation modules for G. This calculation is a remarkable piece of work which suggests the level of difficulty of determining explicitly the simple Mackey functors in general.

Our analogy with Lie theory immediately suggests that it might be possible to gain information about the simple functors from the analogues of the Verma modules. When dealing with a Lie algebra we have available the Weyl character formula and Kazdan-Lusztig theory which express the formal characters of the simple modules and Verma modules in terms of each other. At the moment such precise expressions in the context of Mackey functors seem too much to ask for, but we nevertheless do obtain much information from this approach, including the construction of the simple Mackey functors.

First we deal with the construction of Mackey functors $\Delta_{H,V}$ which play the role of Verma modules. For ordinary Mackey functors this was done in [17] and [18] starting with a module V, taking the 'fixed quotient' Mackey functor and applying inflation and induction of Mackey functors. With globally-defined Mackey functors in a manuscript [23] which is as yet unpublished we use a different approach, but in any case we may characterize the $\Delta_{H,V}$ by their adjoint property, which we now state. For this we first define for each Mackey functor M and group H an $R[W(H)]$-module $\underline{M}(H)$ as follows. When M is an ordinary Mackey functor we put

$$\underline{M}(H) = \bigcap_{K \lneq H} \ker(\iota_K^H)^*$$

where $\iota_K^H : K \hookrightarrow H$ denotes the inclusion of subgroups, and when M is a globally-defined Mackey functor we put

$$\underline{M}(H) = \bigcap_{\substack{\alpha: K \to H \\ \text{mono, not iso}}} \ker \alpha^* \cap \bigcap_{\substack{\beta: H \to K \\ \text{epi, not iso}, \ker\beta \in \mathcal{Y}}} \ker \beta_*.$$

In the case of ordinary Mackey functors $\underline{M}(H)$ is the common kernel of the restriction maps to proper subgroups of H, and we may call it the *restriction kernel* of M at H. In the context of group cohomology this has also been called the *essential elements.* With globally-defined Mackey functors, $\underline{M}(H)$ is the common kernel of all Mackey functor operations which have codomain $M(K)$ where K is a strictly smaller group than H. Using the same notation in both cases, we intend the next two results to apply to both ordinary and globally-defined Mackey functors.

Proposition 3.2 *Given a group H, each of the functors* $\mathrm{Mack}_R \to R[W(H)]$*-mod and* $\mathrm{Mack}_R(G) \to R[W(H)]$*-mod specified by $M \mapsto \underline{M}(H)$ has a left adjoint.*

We will denote images of a $R[W(H)]$-module V under these left adjoints by $\Delta_{H,V}$. It is the left adjoint property which allows us (in a similar (dual) way to [17]) to construct a complete list of simple Mackey functors via the following proposition.

Proposition 3.3 *$\Delta_{H,V}(H) \cong V$, and if $\Delta_{H,V}(J) \neq 0$ then H is a quotient of a subgroup of J. Thus $\Delta_{H,V}$ has a unique highest 'weight space', namely its evaluation at H. Furthermore, if V has a unique simple quotient (as an $R[W(H)]$-module), then $\Delta_{H,V}$ has a unique simple quotient (as a Mackey functor).*

4 Mackey Functors with a Good Filtration

We have been using notation inspired by Lie theory and highest weight categories with the evident caution that with Mackey functors not all of the Lie-theoretic properties can be expected to hold. There are, however, circumstances in which the category of Mackey functors is actually a highest weight category and we start this section by stating when this is the case. After that, we summarize the development of [22], where the idea was to look for properties of Mackey functors in general circumstances which are close to those of a highest weight category. With this goal in mind, we focus attention on the Mackey functors $\Delta_{H,V}$ where V is a p-permutation $R[W(H)]$-module[2], that is, a direct summand of a permutation module. It turns out that these Mackey functors satisfy a vanishing condition on their Ext groups which enables us to copy the constructions from Ringel's paper [15] about modules with a good filtration. We conclude by indicating the special properties of Alperin's weights within this framework.

We now state circumstances in which the category of Mackey functors is a highest weight category, or satisfies a stronger condition than this. In this result we take the simple Mackey functors to be parametrized by weights (H, V) where V is simple, and the partial order on these weights is the one which has previously been given.

Theorem 4.1 **([17], [23])** *Let R be a field of characteristic 0.*

(i) *For each finite group G the category of ordinary Mackey functors* $\mathrm{Mack}_R(G)$ *is semisimple.*
(ii) *In general, the category of globally-defined Mackey functors* Mack_R *is a highest weight category, no matter what the classes $\mathcal{X}$ and $\mathcal{Y}$ are taken to be, except that the condition that every object is the union of its subobjects of finite length fails.*
(iii) *When $\mathcal{X}$ and $\mathcal{Y}$ are both taken to consist of the identity group,* Mack_R *is semisimple.*

Perhaps the most promising aspect of this result is the second statement. The idea here is that even though it may be difficult to describe the simple Mackey functors explicitly by direct calculation, at least it may be possible to take the theoretical view of highest weight categories to obtain some structural information.

We now summarize some results from [22], and for the rest of this section consider only ordinary Mackey functors for a fixed group G. We will take R to be a field of positive characteristic p. Some of the results we are about to state have been proved in the generality that R is a complete local ring with residue field of characteristic p, but we suppress mention of this here, to simplify the exposition. We define $\mathcal{D}$ to be the full subcategory of $\mathrm{Mack}_R(G)$ whose objects are the Mackey functors which have a filtration with factors $\Delta_{H,V}$ where V is a p-permutation $R[W(H)]$-module. Since R is a field, the p-permutation modules we consider are

[2] When we use this terminology, R will be field of characteristic p.

the same thing as trivial source modules, and they may be defined to be modules which are direct summands of permutation modules $R\uparrow_{H_1}^G \oplus \cdots \oplus R\uparrow_{H_n}^G$.

As examples of what functors we expect to find in $\mathcal{D}$, note that every projective $R[W(H)]$-module is a p-permutation module, and so if P is a projective RG-module which is the projective cover of a simple RG-module V, the functor $\Delta_{1,P}$ lies in $\mathcal{D}$ and has a unique simple quotient (namely $S_{1,V}$), by Proposition 3.3. Observe that if (H,V) happens to be one of Alperin's weights (so H is a p-group and V is a simple projective $R[W(H)]$-module) then $\Delta_{H,V}$ lies in $\mathcal{D}$. We should also point out that contrary to the suggestive use of the symbol Δ, not every functor $\Delta_{H,V}$ need have a simple quotient. For an example of this, if V is a p-permutation RG-module which does not have a unique simple quotient, then $\Delta_{1,V}$ does not have a unique simple quotient.

We have defined $\mathcal{D}$ using p-permutation modules for a number of reasons. One of them is that the following result should be true, thereby guaranteeing that $\mathcal{D}$ is reasonably large.

Theorem 4.2 *$\mathcal{D}$ contains all the projective objects of* $\mathrm{Mack}_R(G)$.

Another reason for using this class of modules is the following surprising statement on the vanishing of Ext groups.

Theorem 4.3 *Suppose that V is a p-permutation $R[W(H)]$-module and U is a p-permutation $R[W(K)]$-module, where H and K are subgroups of G. Then* $\mathrm{Ext}^1_{\mu_R(G)}(\Delta_{H,V}, \Delta_{K,U}) = 0$ *unless K is conjugate to a proper subgroup of H.*

The reason one might find this result surprising is that if one takes a pair of p-permutation RG-modules V and U (for example, $V = U = R$), then it certainly need not be the case that $\mathrm{Ext}^1_{RG}(V,U) = 0$, and yet according to the theorem $\mathrm{Ext}^1_{\mu_R(G)}(\Delta_{1,V}, \Delta_{1,U}) = 0$. It is a fact that the assignment $V \mapsto \Delta_{1,V}$ embeds the full subcategory of RG-modules whose objects are p-permutation modules as a full subcategory of $\mathcal{D}$. We see that this embedding functor kills Ext groups.

The vanishing of Ext groups is exactly what is needed to invoke a theorem of Ringel from [15]. If we list the Mackey functors $\Delta_{H,V}$ (with V a p-permutation module) so that $|H|$ never decreases, then we have a list of Mackey functors $M_1, \ldots, M_n$ so that $\mathrm{Ext}(M_i, M_j) = 0$ unless $i > j$. We immediately deduce from Ringel's theorem:

Theorem 4.4 *$\mathcal{D}$ is functorially finite in* $\mathrm{Mack}_R(G)$.

We are also able to show that $\mathcal{D}$ is closed under taking direct summands, and by construction it is closed under taking extensions. This is what is needed to be able to apply a result of Auslander and Smalø to deduce the following.

Corollary 4.5 *$\mathcal{D}$ has relative almost split sequences.*

It is quite interesting, but laborious, to construct the relative almost split sequences in particular cases. We did this in [22] for the cyclic group of order 4 over the field of two elements, this being the largest example we were able to handle at the time. The result is surprisingly complicated for such a small group, and indicates the richness of the structure which is being considered.

In the situation of modules for a quasi-hereditary algebra, Ringel went on to charaterize the modules which have both a Δ and a ∇ filtration, and showed that they are the Ext-injective objects in the category of modules with a Δ-filtration. Here we say that an object T in $\mathcal{D}$ is Ext-*injective* if $\mathrm{Ext}^1(M,T) = 0$ for all objects

M in $\mathcal{D}$, and Ext-*projective* if it satisfies $\mathrm{Ext}^1(T, M) = 0$ for all objects M in $\mathcal{D}$. In our situation with Mackey functors we define in [22] the notion of an Ext-injective hull of an object of $\mathcal{D}$, and dually the Ext-projective cover. We show that these exist for each object of $\mathcal{D}$, and are unique up to isomorphism. We then give the following characterization of the Ext-injectives and Ext-projectives in $\mathcal{D}$.

Theorem 4.6 *The indecomposable* Ext-*injective objects of* $\mathcal{D}$ *are precisely the* Ext-*injective hulls* $T_{H,V}$ *of the functors* $\Delta_{H,V}$ *as* H *ranges over subgroups of* G *and* V *ranges over indecomposable* p*-permutation modules for* $R[W(H)]$*, and the indecomposable* Ext-*projective objects of* $\mathcal{D}$ *are the* Ext-*projective covers of the* $\Delta_{H,V}$.

We continue to follow Ringel's development in [15] and consider the endomorphism ring of the direct sum of all the indecomposable Ext-injective objects in $\mathcal{D}$.

Theorem 4.7 $\mathrm{End}(\bigoplus_{H,V} T_{H,V})$ *is an algebra which is standardly stratified in the sense of* [8].

Quasi-hereditary algebras are standardly stratified, but not conversely in general, and in fact in the case of Mackey functors for C_4 over the field of two elements the algebra constructed in this theorem is not quasi-hereditary.

We conclude by mentioning the special categorical properties of the functors $\Delta_{H,V}$ when the pair (H, V) is a weight in Alperin's sense.

Proposition 4.8 *Let* R *be a field of characteristic* p.

(i) (H, V) *is a weight in the sense of Alperin if and only if the Mackey functor* $\Delta_{H,V}$ *is simple.*

(ii) *A Mackey functor* $\Delta_{H,V}$ *is* Ext-*injective in* $\mathcal{D}$ *if and only if* V *is a projective* $R[W(H)]$*-module. Thus if* (H, V) *is a weight in the sense of Alperin then* $\Delta_{H,V}$ *is* Ext-*injective.*

(iii) *A Mackey functor* $\Delta_{H,V}$ *is* Ext-*projective in* $\mathcal{D}$ *if and only if* $H = O^p(H)$*, that is,* H *has no non-identity* p*-group as a quotient.*

The Ext-injective objects are precisely the ones which occur at the right end of the relative Auslander-Reiten quiver of $\mathcal{D}$ (provided we maintain the convention that arrows are drawn from left to right) since these are exactly the objects which are not the start of a relative almost split sequence. By (i), the $\Delta_{H,V}$ where (H, V) is a weight in Alperin's sense are among these. On the other hand, the Ext-projective objects are precisely the ones which occur at the left end of the relative Auslander-Reiten quiver of $\mathcal{D}$. By (iii), these include the $\Delta_{1,V}$ where V is the projective cover of a simple RG-module. Alperin's conjecture asserts that these two sets of functors, which appear among those at the left and right ends of the relative Auslander-Reiten quiver of $\mathcal{D}$, have the same size. It remains an open question whether or not the structure we have just been describing can shed any light on the equality of sizes of these two sets.

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