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BOUNDING THE RANKS OF ZG-LATTICES BY THEIR RESTRICTIONS TO ELEMENTARY ABELIAN SUBGROUPS

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1. Introduction

A number of authors have shown how the homological properties of modules for the group ring of a finite group G over a field k of characteristic p are in some sense controlled by the elementary abelian p-subgroups of G. Quillen proved in [8] that the Krull dimension of the mod p cohomology ring of G is the maximum rank of an elementary abelian p-subgroup of G. Subsequently Chouinard proved in [4] that a kG-module is projective if and only if it is projective on restriction to all elementary abelian p-subgroups of G, and more recently Alperin and Evens [2] and Carlson [3] have proved theorems in characteristic p which imply the results of Quillen and Chouinard as corollaries. In this paper we consider the problem of obtaining analogous results to these for the integral group ring ZG, and when we do this the rôle of the elementary abelian p-subgroups will be taken by the set of all such subgroups for all prime divisors p of $|G|$. Our main result is an integral version of Carlson's theorem and this implies integral versions of the theorems of Alperin–Evens and Chouinard.

If M is any ZG-lattice, that is a ZG-module which is a free abelian group of finite rank, we may write $M = \mathrm{core}(M) \oplus \mathrm{proj}(M)$ where $\mathrm{proj}(M)$ is a projective module and $\mathrm{core}(M)$ is a submodule of M with no projective summands. In general the submodule $\mathrm{core}(M)$ is not uniquely determined by these requirements, but all the possible choices for $\mathrm{core}(M)$ will have the same rank as an abelian group. We will prove:

Theorem. *Let G be a finite group. There exists a constant B with the property that if M is any ZG-lattice with no nonzero projective summands, there exists an elementary abelian p-subgroup E of G for some prime p such that*

$$\mathrm{rank}_Z(M) \leq B \cdot \mathrm{rank}_Z(\mathrm{core}\, M{\downarrow_E}).$$

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The theorem includes a number of results as corollaries. The following integral version of Chouinard's theorem is an immediate application:

Corollary A. *The ZG-lattice M is projective if and only if it is projective on restriction to every elementary abelian p-subgroup of G, for every prime divisor p of* $|G|$.

However, one can also deduce this result rather easily from Chouinard's original theorem [4] using, for example, [5, 3.13 and 8.6].

The second corollary is due in substance to Talelli [10] and concerns periodicity under the Heller operator. The ZG-lattice M is said to be *periodic* if there is a nontrivial exact sequence of modules

$$0 \to M \to A_n \to \cdots \to A_0 \to M \to 0 \tag{1}$$

in which the A_i are finitely generated projective ZG-modules.

Corollary B. *The following are equivalent for the ZG-lattice M:*

(i) *M is periodic.*

(ii) *M is periodic on restriction to every elementary abelian p-subgroup of G, for each prime* $p \mid |G|$.

(iii) *For every prime* $p \mid |G|$, *M/pM is periodic as a* $(Z/pZ)E$*-module for every elementary abelian p-subgroup E of G.*

Corollaries A and B are both special cases of the next result, which is an integral version of the theorem of Alperin and Evens [2]. Their theorem is stated in terms of the 'complexity' of modules in characteristic p, and we mimic their definition of complexity as follows in the case of ZG-lattices. Given a ZG-lattice M let

$$\begin{array}{ccccccccc} \cdots \to & P_2 & \to & P_1 & \to & P_0 & \to & M & \to 0 \\ & \searrow & \nearrow & \searrow & \nearrow & & & & \\ & & K_1 & & K_0 & & & & \end{array} \tag{2}$$

be a minimal projective resolution of M. By this we mean that for each n, P_n is a projective of minimal rank subject to having K_{n-1} as an image. We will say that the complexity of M is c provided that c is the least nonnegative integer such that there is a positive number λ with $\operatorname{rank}_Z(P_d) \leq \lambda \cdot d^{c-1}$ for all sufficiently large d, and we will write $c_{ZG}(M)$ for c. It will be shown in Section 3 that every ZG-lattice has a complexity. We can prove:

Corollary C. *Let M be a ZG-lattice. Then there exists a prime* $p \mid |G|$ *and an elementary abelian p-subgroup E of G such that* $c_{ZG}(M) = c_{ZE}(M)$.

In fact we will also show that if p is the prime in Corollary C, then $c_{ZG}(M)$ equals the complexity of M/pM as a $(Z/pZ)G$-module in the sense of Alperin and Evens.

Throughout this paper we will use $\hat{Z}_{(p)}$ to denote the ring of p-adic integers, and if M is a ZG-lattice we will write $M_{(p)} = \hat{Z}_{(p)} \otimes_Z M$. We also wish to extend in an obvious way the notation for the decomposition $M = \mathrm{core}(M) \oplus \mathrm{proj}(M)$ to apply in the case that M is a finitely generated RG-module, where R may be either Z, $\hat{Z}_{(p)}$ or a field.

2. Proof of the theorem

There are two stages to the proof. In the first stage we produce the prime p in the statement of the theorem and reduce the problem about the ZG-lattice M to a problem concerning $M_{(p)}$. The second stage is to prove the analogous statement to the theorem for $M_{(p)}$, and we can deduce this fairly easily from Carlson's theorem [3]. If M is any ZG-lattice let $\sigma_p(M) = \mathrm{rank}_{\hat{Z}_{(p)}}(\mathrm{core}\, M_{(p)})$, and put $\sigma(M) = \max\{\sigma_p(M) : p \mid |G|\}$.

Lemma 2.1. *There exists a constant B_1 such that whenever M is a ZG-lattice with no projective summands, then* $\mathrm{rank}_Z(M) \leq B_1 \cdot \sigma(M)$.

Proof. Let χ be the ordinary character of M. If x is any element of G of order divisible by a prime p, then $|\chi(x)| \leq \sigma_p(M)$; for if χ_1, χ_2 are the characters of $\mathrm{core}\, M_{(p)}$ and $\mathrm{proj}\, M_{(p)}$, then $\chi = \chi_1 + \chi_2$, $\chi_2(x) = 0$ and $|\chi_1(x)| \leq |\chi_1(1)| = \sigma_p(M)$. Thus for all non-identity x in G, $|\chi(x)| \leq \sigma(M)$.

A ZG-lattice N has a nonzero projective summand if and only if $\hat{Z}_{(p)}G$ is a summand of $N_{(p)}$ for all primes $p \mid |G|$ (for example, by [5, 6.5], since if $\hat{Z}_{(p)}G$ is a summand of $N_{(p)}$, then $(Z/pZ)G$ is a summand of N/pN). Since M has no projective summand there exists a prime $q \mid |G|$ and an indecomposable projective $\hat{Z}_{(q)}G$-lattice which appears as a summand of $M_{(q)}$ a smaller number of times than in $\hat{Z}_{(q)}G$. Let $\eta_1, \ldots, \eta_s$ be the characters of the indecomposable projective $\hat{Z}_{(q)}G$-modules and let $\mu_1, \ldots, \mu_s$ be the integers for which $\mu_1\eta_1 + \cdots + \mu_s\eta_s$ is the character of the regular representation. If $\chi_{\mathrm{proj}} = \lambda_1\eta_1 + \cdots + \lambda_s\eta_s$ is the character of $\mathrm{proj}\, M_{(q)}$, then for some j, $\lambda_j < \mu_j$. Writing χ_{core} for the character of $\mathrm{core}\, M_{(q)}$, we have that for any nonidentity element x of G,

$$\left| \sum_{i=1}^{s} \lambda_i \eta_i(x) \right| = |\chi_{\mathrm{proj}}(x)| \leq |\chi(x)| + |\chi_{\mathrm{core}}(x)| \leq \sigma(M) + \sigma_q(M) \leq 2\sigma(M).$$

That is

$$\sigma(M)^{-1} \left| \sum_{i=1}^{s} \lambda_i \eta_i(x) \right| \leq 2 \quad \text{for all nonidentity } x \text{ in } G, \tag{3}$$

since $\sigma(M) \neq 0$ if, as we may suppose, $M \neq 0$.

Let $1 = x_1, x_2, \ldots, x_r$ be representatives of the q-regular conjugacy classes of G, and let H be the $s \times r$ matrix $(\eta_i(x_j))$. We may regard the entries $\eta_i(x_j)$ as lying in some

algebraic number field K contained in the complex numbers and which is a splitting field for G. If we let R be the integers of K and $\mathfrak{q}$ be a prime of R containing q, then if $\eta_1', \dots, \eta_r'$ are the characters of the indecomposable projective $R_{\mathfrak{q}}G$-modules, the corresponding square matrix $H' = (\eta_i'(x_j))$ is invertible. Each row of H is a linear combination of certain rows of H', and each row of H' appears in the expression for precisely one row of H. It follows that the rank of H is s.

Let $e_1, \dots, e_r$ be a standard basis for the vector space K^r, let T be the line e_1K, and let

$$U = \{(y_1, \dots, y_r) \in K^r : y_1 = 0,\ |y_i| \leq 2 \text{ for all } i\}.$$

Here $|y_i|$ means the absolute value of y_i as a complex number. Then condition (3) may be restated as

$$\sigma(M)^{-1}(\lambda_1, \dots, \lambda_s) \cdot H \in T + U = \{t + u : t \in T, u \in U\}.$$

Now U is a bounded set, so the distance of any point in $T + U$ from the line T is at most $\sup\{\|u\| : u \in U\}$. T is the image under H of the line $(\mu_1, \dots, \mu_s)K$, so since H is both a monomorphism and a linear map any point in the preimage of $T + U$ under H lies a bounded distance from $(\mu_1, \dots, \mu_s)K$. Let d be a bound for this distance and let $c \in K$ be a number for which the distance between $\sigma(M)^{-1}(\lambda_1, \dots, \lambda_s)$ and $c(\mu_1, \dots, \mu_s)$ is at most d. It follows that $|c\mu_j - \sigma(M)^{-1}\lambda_j| \leq d$ for each j. Since we also know that $0 \leq \lambda_j < \mu_j$ for some j, and $\sigma(M) \geq 1$ we have $0 \leq \mu_j - \sigma(M)^{-1}\lambda_j \leq \mu_j$, and so $|c\mu_j - \mu_j| \leq d + \mu_j$. Thus

$$|c| \leq 2 + d\mu_j^{-1} \leq 2 + \max\{\mu_1^{-1}, \dots, \mu_s^{-1}\} \cdot d$$

and this latter number depends only on the prime q. This bound for $|c|$ means that $\sigma(M)^{-1}(\lambda_1, \dots, \lambda_s)$ lies in a bounded region of K^s, and there is a constant b_q such that

$$\max\{\lambda_1, \dots, \lambda_s\} \leq b_q \cdot \sigma(M).$$

We will put $f_q = \eta_1(1) + \cdots + \eta_s(1)$, and similarly we define constants b_p and f_p for all the prime divisors p of $|G|$. Let

$$B_1 = 1 + \max\{b_p \cdot f_p : p \mid |G|\}.$$

Then

$$\begin{aligned} \operatorname{rank}_{\mathbb{Z}}(M) &= \sigma_q(M) + \sum_{i=1}^{s} \lambda_i \eta_i(1) \\ &\leq \sigma_q(M) + \max\{\lambda_1, \dots, \lambda_s\} \cdot \sum_{i=1}^{s} \eta_i(1) \\ &\leq \sigma(M) \cdot (1 + b_q \cdot f_q) \leq B_1 \cdot \sigma(M). \end{aligned}$$

Lemma 2.2. *For each prime p there exists a constant $B_2^{(p)}$ with the following property: whenever M is a $\hat{Z}_{(p)}G$-lattice with no nonzero projective summands there is an elementary abelian p-subgroup E of G for which*

$$\operatorname{rank}_{\hat{Z}_{(p)}}(M) \leq B_2^{(p)} \cdot \operatorname{rank}_{\hat{Z}_{(p)}}(\operatorname{core} M\downarrow_E).$$

Proof. It follows from the fact that M has no projective summands that M/pM also has no projective summands (since projective summands over Z/pZ can be lifted to projective summands over $\hat{Z}_{(p)}$). Hence using the theorem of Carlson [3],

$$\operatorname{rank}(M) = \dim M/pM \leq B_2^{(p)} \cdot \dim(\operatorname{core}(M/pM)\downarrow_E)$$

for some elementary abelian p-subgroup E of G and some constant $B_2^{(p)}$, where the dimensions are taken over Z/pZ. Also, since $\operatorname{core}((M/pM)\downarrow_E)$ is the reduction mod p of $\operatorname{core}(M\downarrow_E)$ we obtain $\operatorname{rank}(M) \leq B_2^{(p)} \cdot \operatorname{rank}(\operatorname{core} M\downarrow_E)$ as required.

We now complete the proof of the theorem. Given the ZG-lattice M choose a prime $p \mid |G|$ for which $\sigma(M) = \sigma_p(M)$, and choose an elementary abelian p-subgroup E of G for which

$$\sigma_p(M) \leq B_2^{(p)} \cdot \operatorname{rank}_{\hat{Z}_{(p)}} \operatorname{core}(M_{(p)}\downarrow_E),$$

by Lemma 2.2. Notice that

$$\operatorname{rank}_{\hat{Z}_{(p)}} \operatorname{core}(M_{(p)}\downarrow_E) = \operatorname{rank}_Z \operatorname{core} M\downarrow_E,$$

since E is a p-group (for example, by [5, 6.5]). If we let $B = B_1 \cdot \max\{B_2^{(p)} : p \mid |G|\}$, then

$$\operatorname{rank}_Z(M) \leq B_1 \cdot \sigma(M) \leq B_1 \cdot B_2^{(p)} \cdot \operatorname{rank}_Z \operatorname{core}(M\downarrow_E) \leq B \cdot \operatorname{rank}_Z \operatorname{core}(M\downarrow_E).$$

3. Proofs of Corollaries B and C

Before proving the corollaries we outline some properties of the resolution (2) of the ZG-lattice M. In [9], Swan gave a method for deducing the ranks of the projective modules P_d in (2) from information about resolutions of M/pM for the various primes $p \mid |G|$, and we may describe this method as follows. We will define the dth partial Euler characteristic of the minimal resolution (2) to be

$$v_d(G, M) = \sum_{j=0}^{d} (-1)^{d-j} \operatorname{rank}_Z(P_j \otimes_{ZG} Z).$$

It is a consequence of the methods of [9] that if we had used any other minimal ZG-resolution of M to calculate $v_d(G, M)$ we would have obtained the same answer. If S is any simple kG-module where $k = Z/pZ$ for some prime $p \mid |G|$ we will set

$$f_d(S) = \sum_{j=0}^{d} (-1)^{d-j} \dim_k \operatorname{Ext}_{kG}^j(k \otimes_Z M, S).$$

Then $v_d(G, M)$ is related to the numbers $f_d(S)$ by the formula

$$v_d(G, M) = \max\{\lfloor f_d(S)/\dim_k S \rfloor : S \text{ is a simple } kG\text{-module}, \\ k = Z/pZ \text{ and } p \mid |G|\} \quad (4)$$

where in taking the maximum, p is allowed to range over all prime divisors of $|G|$, and the special brackets indicate the least integer greater than the number they enclose. This formula is due to Swan in [9, Proposition 6.1] for the case $M = Z$, but it is quite easy to extend his arguments to deal with arbitrary lattices M. See also [6] for a further discussion of this formula.

In [7] Higman calls a function $\theta: Z \to Z$ a PORC function if there is an integer m such that for each r with $1 \leq r \leq m$, $\theta(mn + r)$ is a polynomial in n. We will call θ an almost PORC function if θ is PORC except at finitely many integers. It is well known (see [1]) that with the previous notation, $\dim_k \mathrm{Ext}^j_{kG}(k \otimes_Z M, S)$ is an almost PORC function of j. Hence $f_d(S)$ is an almost PORC function of d, since it is obtained as an alternating sum of almost PORC functions. It is not hard to see from (4) that $v_d(G,M)$ is also an almost PORC function of d. In the definition of $v_d(G,M)$, $\mathrm{rank}_Z(P_d \otimes_{ZG} Z)$ is really just the projective rank of P_d, so using Swan's structure theorem for projectives [5, 4.8] we have

$$\mathrm{rank}_Z(P_d) = |G| \cdot \mathrm{rank}_Z(P_d \otimes_{ZG} Z) = |G| \cdot (v_d(G,M) + v_{d-1}(G,M))$$

and $\mathrm{rank}_Z(P_d)$ is an almost PORC function of d. This shows that the complexity $c_{ZG}(M)$ always exists when M is a ZG-lattice.

Suppose that $\mathrm{rank}_Z(P_d)$ is almost PORC with respect to the integer m, so that for $1 \leq r \leq m$ there are polynomials h_r with $h_r(n) = \mathrm{rank}_Z(P_{mn+r})$ except at finitely many values. We will need to use the fact that the degrees of these polynomials are all the same. This is because $\mathrm{rank}_Z(P_{d+1}) \leq |G| \cdot \mathrm{rank}_Z(P_d)$ always holds, so that $\deg h_r \leq \deg h_{r-1}$ for $2 \leq r \leq m$ and $\deg h_1 \leq \deg h_m$. Hence the degrees are all equal.

Proof of Corollary C. For each kernel K_i in the minimal ZG-resolution (2) of M there is a prime $p_i \mid |G|$ and an elementary abelian p_i-subgroup E_i of G for which $\mathrm{rank}\, K_i \leq B \cdot \mathrm{rank}(\mathrm{core}(K_i{\downarrow_{E_i}}))$ where B is the constant in the main theorem. Since G has only finitely many subgroups there is a prime p and an elementary abelian p-subgroup E which appears infinitely often in the sequence of E_i. Writing $K_i{\downarrow_E} = L_i \oplus A_i$ where $L_i = \mathrm{core}(K_i{\downarrow_E})$ and $A_i = \mathrm{proj}(K_i{\downarrow_E})$ we may obtain a minimal projective ZE-resolution of $\mathrm{core}(M{\downarrow_E})$

$$\begin{array}{ccccccccc} \cdots \to & Q_2 & \to & Q_1 & \to & Q_0 & \to & \mathrm{core}(M{\downarrow_E}) & \to 0 \\ & \searrow\ \nearrow & & \searrow\ \nearrow & & & & & \\ & L_1 & & L_0 & & & & & \end{array} \tag{5}$$

with the property that

$$\begin{array}{ccccccccccccc} \cdots \to & Q_2 \oplus A_2 \oplus A_1 & \to & Q_1 \oplus A_1 \oplus A_0 & \to & Q_0 \oplus A_0 \oplus \mathrm{proj}(M{\downarrow_E}) & \to & M{\downarrow_E} & \to 0 \\ & \searrow\ \nearrow & & & & \searrow\ \nearrow & & & \\ & K_1 & & & & K_0 & & & \end{array}$$

is the restriction of the original resolution (2) to E. Now $\mathrm{rank}\, P_{i+1} \leq |G| \cdot \mathrm{rank}\, K_i$ and $\mathrm{rank}\, L_i \leq \mathrm{rank}\, Q_i$ so since $\mathrm{rank}\, K_i \leq B \cdot \mathrm{rank}\, L_i$ holds for infinitely many i, we have $\mathrm{rank}\, P_{i+1} \leq B \cdot |G| \cdot \mathrm{rank}\, Q_i$ for infinitely many i. But the polynomials giving

rank P_{i+1} all have the same degree, as do the polynomials giving rank Q_i, so we may deduce the existence of a constant λ such that rank $P_i \leq \lambda \cdot$ rank Q_i for all sufficiently large i. Hence $c_{ZG}(M) \leq c_{ZE}(M)$. It is easy to see that $c_{ZE}(M) \leq c_{ZG}(M)$, since rank $Q_i \leq$ rank P_i for all i. Thus $c_{ZG}(M) = c_{ZE}(M)$.

We now justify the remark after the statement of Corollary C that $c_{ZG}(M)$ equals the complexity of $k \otimes_Z M$ in the sense of Alperin [1], where $k = Z/pZ$ and p is the prime in Corollary C. This complexity is defined as follows: we choose a minimal kG-projective resolution

$$\cdots \to U_2 \to U_1 \to U_0 \to k \otimes_Z M \to 0.$$

Then $c_{kG}(k \otimes_Z M)$ is the smallest nonnegative integer c for which there exists a positive constant λ with $\dim_k(U_d) \leq \lambda \cdot d^{c-1}$ for all sufficiently large d.

Proposition 3.1. *Let M be a ZG-lattice and let p be the prime in the statement of Corollary C for which $c_{ZG}(M) = c_{ZE}(M)$ for some elementary abelian p-subgroup E of G. Then $c_{ZG}(M) = c_{kG}(k \otimes_Z M)$.*

Proof. Any ZE-lattice N will have a nonzero ZE-projective summand if and only if $k \otimes_Z N$ has a nonzero kE-projective summand (by [5, 6.3 and 6.4]). Thus, by tensoring the minimal ZE-resolution (5) with k we obtain a minimal kE-resolution of $k \otimes_Z M$. Therefore $c_{ZE}(M) = c_{kE}(k \otimes_Z M)$. But since $c_{kE}(k \otimes_Z M) \leq c_{kG}(k \otimes_Z M) \leq c_{ZG}(M)$, we have $c_{ZG}(M) = c_{kG}(k \otimes_Z M)$.

Proof of Corollary B. Corollary B is really the statement of Corollary C (together with Proposition 3.1) in the case of complexity at most 1. To see this, observe first that $c_{ZG}(M) = 0$ if and only if M has a finite projective resolution, if and only if M is projective. As was observed in [2], periodicity of $k \otimes_Z M$ as a kG-module, where $k = Z/pZ$, is equivalent to $c_{kG}(k \otimes_Z M) \leq 1$. Thus the proof of Corollary B will be complete once we have established:

Lemma 3.2. *A nonprojective ZG-lattice M is periodic if and only if $c_{ZG}(M) = 1$.*

Proof. If M is periodic we may construct a projective resolution of M in which all the modules have bounded rank, simply by linking together copies of the exact sequence (1), and so $c_{ZG}(M) \leq 1$.

Conversely if $c_{ZG}(M) = 1$ there exists a projective resolution

$$\begin{array}{ccccccccc} \cdots & \to & P_2 & \to & P_1 & \to & P_0 & \to & M \to 0 \\ & & \searrow & \nearrow & \searrow & \nearrow & & & \\ & & & K_1 & & K_0 & & & \end{array}$$

for which $\operatorname{rank}_Z(P_d)$ is bounded as a function of d, and we might as well suppose this is a minimal resolution of M. Then $\operatorname{rank}_Z(K_d)$ is bounded also, so by the

Jordan–Zassenhaus theorem there exist integers $i<j$ with $K_i \cong K_j$, and if $K_i = M$ we are done. In any case we claim that provided $i>0$, K_{i-1} and K_{j-1} are locally isomorphic. For, dualizing the resolution and applying Schanuel's lemma to

$$0 \to K_{i-1}^* \to P_i^* \to K_i^* \to 0,$$

$$0 \to K_{j-1}^* \to P_j^* \to K_j^* \to 0,$$

we have $K_{i-1}^* \oplus P_j^* \cong K_{j-1}^* \oplus P_i^*$. By minimality of the resolution and local cancellation [5, 4.4] we deduce that K_{i-1}^* and K_{j-1}^* are locally isomorphic, and hence K_{i-1} and K_{j-1} are also. Proceeding inductively, M is locally isomorphic to K_{j-i-1} and we may embed K_{j-i-1} in M so that the quotient has exponent prime to $|G|$ [5, 4.2]. Forming the pushout

$$\begin{array}{ccc} K_{j-i-1} & \longrightarrow & P_{j-i-1} \\ \downarrow & & \downarrow \\ M & \longrightarrow & Q \end{array}$$

we obtain a short exact sequence $0 \to M \to Q \to K_{j-i-2} \to 0$ in which Q is projective, since P_{j-i-1} is embedded in it with a quotient of exponent prime to $|G|$. Inserting this sequence into the original resolution gives an exact sequence which demonstrates the periodicity of M.

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