RESTRICTING $\mathbb{Z}G$-LATTICES TO ELEMENTARY ABELIAN SUBGROUPS

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1.

In recent years a number of results in modular representation theory have been inspired by the Quillen-Venkov proof in [QV] that the Krull dimension of the mod p cohomology ring of a finite group G equals the maximum rank of an elementary abelian p-subgroup of G. These results are due to Chouinard [Ch], Alperin and Evens [AE] and Carlson [Ca]. The strongest of these results is Carlson's, and since it is also one of the easiest to state, we will do so here. We use the notation that k is a field of characteristic $p > 0$, with $p||G|$, and if M is a kG-module we write $M = \mathrm{core}(M) \oplus \mathrm{proj}(M)$ where proj(M) is projective and core(M) has no non-zero projective summands.

Theorem 0 (J. Carlson [Ca]) Let G be a finite group. There exists a constant C with the property that if M is any finitely generated kG-module with no non-zero projective summands, there exists an elementary abelian p-subgroup E of G such that

$$\dim_k M \leq C.\dim_k \mathrm{core}(M\downarrow_E)$$

It is not difficult to formulate versions of the above results for the integral group ring $\mathbb{Z}G$, and our aim in this article is to show that all of these integral versions are true. There are two points to note in translating the modular statements to the integers. Firstly, one must restrict attention to $\mathbb{Z}G$-lattices and use the $\mathbb{Z}$-rank of a lattice instead of its dimension. The second and more important point is that in removing the dependence on some particular prime one must state one's theorem in terms of the collection of all elementary abelian p-subgroups for all prime divisors p of $|G|$. Thus our main theorem is the following:

Theorem 1 Let G be a finite group. There exists a constant B with the property that if M is any $\mathbb{Z}G$-lattice with no non-zero projective summands, there exists an elementary abelian p-subgroup E of G for some prime p such that

$$\mathrm{rank}_{\mathbb{Z}}(M) \leq B.\mathrm{rank}_{\mathbb{Z}}(\mathrm{core}(M{\downarrow}_E)) .$$

We are now using the notation $M = \mathrm{core}(M) \oplus \mathrm{proj}(M)$ when M is a lattice, and although the choice of the sublattice core(M) may not be unique in this case, its rank is uniquely determined by M.

There is an integral version of the theorem of Alperin and Evens [AE] which is implied by Theorem 1. To state this we must first define the complexity of a $\mathbb{Z}G$-lattice M. Following the definition given by Alperin [A] in the case of a kG-module, let

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0 \qquad (1)$$

be a minimal $\mathbb{Z}G$-projective resolution of M. This means that each projective module P_d has minimal rank subject to having K_{d-1} as an image, or equivalently that the K_d have no non-zero projective summands (see [Sw]). Then we say that the complexity of M is c provided that c is the least non-negative integer such that there exists $\lambda > 0$ with $\mathrm{rank}_{\mathbb{Z}}(P_d) \leq \lambda.d^{c-1}$ for all sufficiently large d. We will write $c_{\mathbb{Z}G}(M)$ for c to distinguish it from the complexity $c_{kG}(k \otimes_{\mathbb{Z}} M)$ of the kG-module $k \otimes_{\mathbb{Z}} M$ as defined by Alperin [A]. The method Swan used to prove Proposition 6.1 in [Sw] allows one to deduce the ranks of the P_d from modular information, and it is possible to show by this method that the complexity $c_{\mathbb{Z}G}(M)$ exists, given that $c_{kG}(k \otimes M)$ always exists.

Corollary 2 Let M be a $\mathbb{Z}G$-lattice. Then there exists a prime $p \mid |G|$ and an elementary abelian p-subgroup E of G such that $c_{\mathbb{Z}G}(M) = c_{\mathbb{Z}E}(M) = c_{kG}(k \otimes M)$ where $k = \mathbb{Z}/p\mathbb{Z}$.

In section 2 we will show how this Corollary may be deduced from Theorem 1, but it can also be proved in a different manner by building on Swan's ideas in [Sw]. It is interesting that both this alternative proof and the proof we shall give of Theorem 1 seem to need the use of the non-singularity of the Cartan matrix at one stage.

Just as in the modular case described in [AE], lattices of complexity 0 and 1 are respectively projective and periodic. In the case of complexity 0, Corollary 2 becomes the following integral version of Chouinard's theorem [Ch]:

Corollary 3 The $\mathbb{Z}G$-lattice M is projective if and only if it is projective on restriction to all elementary abelian p-subgroups of G, for all $p||G|$.

For complexity 1 we recover the following result which is essentially due to Olympia Talelli [T]:

Corollary 4 The following are equivalent for the $\mathbb{Z}G$-lattice M:
(i) M is periodic
(ii) M is periodic on restriction to every elementary abelian p-subgroup of G, for each prime $p||G|$
(iii) For every prime $p||G|$, M/pM is periodic as a $(\mathbb{Z}/p\mathbb{Z})E$-module for every elementary abelian p-subgroup E of G.

Before passing to the proofs of these results we should remark that in particular cases they can be deduced without difficulty from the corresponding modular results just by applying standard lifting techniques. This is so if we restrict G always to be a p-group, or alternatively if we wish to prove analogous results for $\mathbb{Z}_{(p)}G$ where $\mathbb{Z}_{(p)}$ is the ring of p-adic integers and G may be any finite group. Thus we can prove a p-adic version of our main theorem by means of the observation that a $\mathbb{Z}_{(p)}G$-lattice M has a projective summand if and only if M/pM has a projective summand. So if M has no projective summands then

$$\begin{aligned} \operatorname{rank}(M) = \dim(M/pM) &\leq C.\dim \operatorname{core}((M/pM)\downarrow_E) \\ &= C.\operatorname{rank}(\operatorname{core}(M\downarrow_E)) \end{aligned} \tag{2}$$

for some elementary abelian p-subgroup E of G, where C is the constant in Carlson's theorem. The real problem in proving Theorem 1 is that of linking these p-adic statements together to give a global one. Note, however, that in all cases Corollary 3 may be deduced directly from Chouinard's original theorem [Ch], since a $\mathbb{Z}G$-lattice M is projective if and only if M/pM is a projective $(\mathbb{Z}/p\mathbb{Z})G$-module for all prime divisors p of |G| (see 3.13 and 8.6 of [G]). Also, one may prove results such as the following:

all lattices in a block of $\mathbb{Z}_{(p)}G$ with cyclic defect group are periodic.

This is just the combination of the corresponding statement over $\mathbb{Z}/p\mathbb{Z}$ with Proposition 2.2 of [T].

2. Proofs of Theorem 1 and Corollary 2

If M is a $\mathbb{Z}G$-lattice we let $M_{(p)} = \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} M$ and put $\sigma_p(M) = \text{rank}_{\mathbb{Z}_{(p)}}(\text{core } M_{(p)})$, $\sigma(M) = \max\{\sigma_p(M) : p||G|\}$. The following result will allow us to deduce Theorem 1 from the local information:

Proposition 5 There exists a constant B_1 such that whenever M is a $\mathbb{Z}G$-lattice with no projective summands then $\text{rank}_{\mathbb{Z}}(M) \leq B_1.\sigma(M)$.

Proof We produce some inequalities on the values of the character χ of M. Firstly, if x is any non-identity element of G and p is a prime dividing the order of x then $\chi = \chi_1 + \chi_2$ where χ_1 and χ_2 are the characters of $\text{core}(M_{(p)})$ and $\text{proj}(M_{(p)})$ respectively. Because $\chi_2(x) = 0$ (Theorem 4 of [Sw2] is a convenient reference) $|\chi(x)| = |\chi_1(x)| \leq |\chi_1(1)| = \sigma_p(M)$, and so $|\chi(x)| \leq \sigma(M)$ holds for all non-identity x in G. Here we are regarding the values of the characters as lying in some subfield K of the complex numbers which is a splitting field for G, and the absolute value is the usual one for complex numbers.

If $\eta_1,\ldots,\eta_s$ are the characters of the indecomposable projective $\mathbb{Z}_{(p)}G$-modules we may write $\chi_2 = \lambda_1\eta_1 + \ldots + \lambda_s\eta_s$ for some $\lambda_i \in \mathbb{Z}$. For any non-identity element $x \in G$,

$$|\chi_2(x)| \leq |\chi(x)| + |\chi_1(x)| \leq \sigma(M) + \sigma_p(M) \leq 2.\sigma(M).$$

Hence

$$\sigma(M)^{-1}.\left|\sum_{i=1}^{s} \lambda_i\eta_i(x)\right| \leq 2$$

for non-identity x, since $\sigma(M) \neq 0$ if, as we may suppose, $M \neq 0$. If we let $1 = x_1, \ldots, x_r$ be representatives of the p-regular conjugacy classes of G we may interpret this inequality as saying that the vector

$\sigma(M)^{-1}(\lambda_1, \ldots, \lambda_s).H$ lies a bounded distance from the line $(1,0, \ldots, 0)K$ in an r-dimensional K-vector space, where $H = (\eta_i(x_j))$. The $s \times r$ matrix H has rank s (c.f. p.599 of [CR]), and this means that $\sigma(M)^{-1}(\lambda_1, \ldots, \lambda_s)$ lies a bounded distance from the inverse image of $(1,0, \ldots, 0)K$ under H, i.e. from the line $(\mu_1, \ldots, \mu_s)K$ where $\mu_1\eta_1 + \ldots + \mu_s\eta_s$ is the character of the regular representation.

If $\mathrm{proj}(M_{(p)})$ were to contain a copy of the regular representation for every prime divisor p of $|G|$ then M would have a non-trivial projective summand (c.f. 6.3 and 6.4 of [G]). Since M has no non-trivial projective summand there is a prime $p||G|$ with $\lambda_j < \mu_j$ for some j. This extra condition means that $\sigma(M)^{-1}(\lambda_1, \ldots, \lambda_s)$ actually lies in a bounded region of K^s for this p, and moreover the bound is independent of the module M. It is now easy to deduce that

$$\begin{aligned}\mathrm{rank}(M) = \chi_1(1) + \chi_2(1) &= \sigma_p(M) + \sum_{i=1}^{s} \lambda_i\eta_i(1) \\ &\leq \sigma(M) + \max\{\lambda_1, \ldots, \lambda_s\}.\sum_{i=1}^{s} \eta_i(1) \\ &\leq B_1.\sigma(M)\end{aligned}$$

for some constant B_1 which depends only on G.

Proof of Theorem 1 Choose a prime p so that $\sigma(M) = \sigma_p(M)$. By Proposition 5 and inequality (2) there exists an elementary abelian p-subgroup E of G for which

$$\begin{aligned}\mathrm{rank}_{\mathbb{Z}}(M) \leq B_1.\sigma(M) &\leq B_1.C.\mathrm{rank}_{\mathbb{Z}_{(p)}} \mathrm{core}(M_{(p)}\downarrow_E) \\ &= B_1.C.\mathrm{rank}_{\mathbb{Z}}\mathrm{core}(M\downarrow_E),\end{aligned}$$

the latter equality holding since E is a p-group (c.f. 6.3 and 6.4 of [G]). Now take $B = B_1.C$.

Proof of Corollary 2 Minimality of the resolution (1) is equivalent to the requirement that each kernel K_d has no non-zero projective summand. Furthermore, whenever H is a subgroup of G, $\mathrm{core}(K_d\downarrow_H)$ is the d-th kernel in a minimal $\mathbb{Z}H$-projective resolution of $M\downarrow_H$ and thus $c_{\mathbb{Z}G}(M) \geq c_{\mathbb{Z}H}(M)$ always.

On the other hand, if $c = \max\{c_{\mathbb{Z}E}(M\downarrow_E)\}$ where the maximum is taken over all elementary abelian p-subgroups E, for all $p||G|$, then for all sufficiently large d,

$$\mathrm{rank}(K_d) \leq B.\mathrm{rank}\,\mathrm{core}(K_d\downarrow_{E_d}) \leq B.\lambda_{E_d}d^{c-1}$$

for some elementary abelian subgroup E_d and constant λ_{E_d}. Since G has

only finitely many subgroups we may choose a $\lambda \geq \lambda_{E_d}$ for all d, and then $\text{rank}(K_d) \leq B.\lambda d^{c-1}$. Therefore

$$\text{rank}(P_d) = \text{rank}(K_d) + \text{rank}(K_{d-1}) \leq 2.B.\lambda.d^{c-1}$$

and $c_{\mathbb{Z}G}(M) \leq c$, as required for the first equality. For the subgroup E we may take any elementary abelian p-subgroup which realizes the maximum value $c = c_{\mathbb{Z}E}(M\downarrow_E)$.

Since E is a p-group, $k \otimes_{\mathbb{Z}} (\text{core}(K_d\downarrow_E))$ has no non-zero projective summands (6.5 of [G]) and so $c_{kE}(k \otimes M) = c_{\mathbb{Z}E}(M)$. Since $c_{kE}(k \otimes M) \leq c_{kG}(k \otimes M) \leq c_{\mathbb{Z}G}(M)$, we have $c_{\mathbb{Z}E}(M) = c_{kG}(k \otimes M) = c_{\mathbb{Z}G}(M)$.

3. An Example

It is not hard to produce indecomposable non-projective lattices M for particular groups G for which the quantity $\text{rank}(M) - \sigma(M)$ can become arbitrarily large. We will illustrate this phenomenon with some lattices which arise in a natural way. Note that by Proposition 5 as $\text{rank}(M) - \sigma(M)$ becomes large so rank(M) must also become large in proportion, and to find such lattices we must deal with a group of infinite lattice type.

To fix our example let $G = A_5 \times A_5$ be the product of two copies of the alternating group on five letters and let

$$\cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

$$K_1 \qquad K_0$$

be a minimal projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Minimality of the resolution means that the K_d have no projective summands, and by Chapter 8 of [G] these modules are indecomposable since the prime graph of G is connected. Now for each prime p, $\text{core}((K_d)_{(p)})$ is the d-th kernel in a minimal $\mathbb{Z}_{(p)}G$-resolution of $\mathbb{Z}_{(p)}$. Since $c_{kG}(k) = 2$ when $k = \mathbb{Z}/p\mathbb{Z}$ for $p = 3$ and 5 and $c_{\mathbb{Z}G}(\mathbb{Z}) = 4$, it follows that $\text{rank}_{\mathbb{Z}}(K_d) - \text{rank core}((K_d)_{(p)})$ becomes arbitrarily large as $d \to \infty$ for $p = 3$ or 5. Modulo 2, A_5 has a non-principal block and hence G has a non-principal block also. It is clear that the only projectives which appear in a minimal $\mathbb{Z}_{(2)}G$-resolution of $\mathbb{Z}_{(2)}$ belong to the principal block, and so for each d the non-principal summands of $(P_d)_{(2)}$ must also appear as summands of either $(K_d)_{(2)}$ or $(K_{d-1})_{(2)}$. By Swan's structure theorem for projectives (4.8 of [G]), $(P_d)_{(2)} \cong \mathbb{Z}_{(2)}G^n$ for some n, so that since $c_{\mathbb{Z}G}(\mathbb{Z}) = 4$ the number of non-principal summands of $(P_d)_{(2)}$

increases without bound as $d \to \infty$. This means we can find a subsequence of the modules K_d so that $\text{rank}(K_d) - \text{rank core}(K_d)_{(2)}$ becomes arbitrarily large, and putting that together with the information about the primes 3 and 5 we have that $\text{rank}(K_d) - \sigma(K_d)$ becomes arbitrarily large.

REFERENCES

[A] J. Alperin, Periodicity in groups, Ill. J. Math. 21 (1977), 776-783

[AE] J. Alperin and L. Evens, Representations, resolutions and Quillen's dimension theorem, to appear.

[Ca] J. F. Carlson, The dimensions of modules and their restrictions over modular group algebras, to appear.

[Ch] L. G. Chouinard, Projectivity and relative projectivity over group rings, J. Pure Appl. Algebra 7 (1976), 278-302

[CR] C. W. Curtis and I. Reiner, Representation theory of finite groups and associative algebras (Wiley Interscience, New York, 1962)

[G] K. W. Gruenberg, Relation modules of finite groups, Amer. Math. Soc. Regional Conf. Ser. 25 (1975)

[QV] D. Quillen and B. B. Venkov, Cohomology of finite groups and elementary abelian subgroups, Topology 11 (1972), 317-318

[Sw] R. G. Swan, Minimal resolutions for finite groups, Topology 4 (1965), 193-208

[Sw2] R. G. Swan, The Grothendieck ring of a finite group, Topology 2 (1963), 85-110

[T] O. Talelli, On cohomological periodicity of $\mathbb{Z}G$-lattices, Math. Z. 169 (1979), 119-126

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