Representations and Cohomology of Categories

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Outline

What is a representation of a category?

Category cohomology and the Schur multiplier

Xu's counterexample

The orbit category and Alperin's weight conjecture

Concluding remarks

Theme

Representations of categories are remarkably like representations of groups!

Categories

Let $\mathcal{C}$ be a small category.
Examples:

- a group
- a poset
- the free category associated to a quiver. The objects are the vertices of the quiver, the morphisms are all possible composable strings of the arrows.

The theory of representations of the above examples is well developed and we do not expect to get more information about them from this general theory. We are more interested in other categories, such as the orbit category associated to a family of subgroups of a group, or the categories which arise with p-local finite groups.

Representations

Let R be a commutative ring with 1. A representation of a category $\mathcal{C}$ over R is a functor $M : \mathcal{C} \to R$-mod.

Straightforward example:
$\mathcal{C}$ is the category with five morphisms $\bullet \longleftarrow \bullet \longrightarrow \bullet$.
A representation is a diagram of modules $B \longleftarrow A \longrightarrow C$.
We may be interested in

- the direct limit of this diagram: the pushout;
- is this operation exact?
- Etc.

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A representation of a category is a diagram of modules.

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Well-studied examples of representations

- When $\mathcal{C}$ is a group we get homomorphism $\mathcal{C} \to \mathrm{End}_R(V)$.
- When $\mathcal{C}$ is a poset we get a module for the incidence algebra.
- When $\mathcal{C}$ is the free category associated to a quiver we get a representation of the quiver.
- When $\mathcal{C} = \bullet \longleftarrow \bullet \longrightarrow \bullet$ its path algebra is

$$\begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

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Further examples

- $\mathcal{C} =$ finite dimensional vector spaces over some field. We get generic representation theory.
- $\mathcal{C} =$ finite sets with bijective morphisms. We get species.
- Various constructions in topology and the cohomology of groups: homotopy colimits, the Quillen category.

Category Algebra

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The category algebra $R\mathcal{C}$ is the free R-module with the morphisms of $\mathcal{C}$ as a basis. We define the product of these basis elements to be composition if possible, zero otherwise. Examples:

- When $\mathcal{C}$ is a group we get the group algebra.
- When $\mathcal{C}$ is a poset we get the incidence algebra.
- When $\mathcal{C}$ is the free category associated to a quiver we get the path algebra of the quiver.

Equivalence of representations and modules

Theorem (B. Mitchell)

Representations are 'the same' as $R\mathcal{C}$-modules, if $\mathcal{C}$ has finitely many objects.

Example:

- When $\mathcal{C}$ is a group, representations are the same as modules for the group algebra.
- When $\mathcal{C}$ is the free category associated to a quiver, representations are the same as modules for the path algebra.

Under this correspondence a representation M corresponds to an $R\mathcal{C}$-module $\bigoplus_{x\in\mathrm{Ob}\mathcal{C}} M(x)$. Natural transformations of functors correspond to module homomorphisms.

Constant functors

For any R-module A we define the constant functor $\underline{A} : \mathcal{C} \to R$-mod to be $\underline{A}(x) = A$ on objects x and $\underline{A}(\alpha) = \mathrm{id}_A$ on morphisms α.
Taking A to be R itself we get the constant functor $\underline{R}$.
Example:

- When $\mathcal{C}$ is a group we get the trivial module R.

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Category cohomology

Theorem (Roos, Gabriel-Zisman)

$\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, \underline{R}) \cong H^*(|\mathcal{C}|, R)$ *where* $|\mathcal{C}|$ *is the nerve of* $\mathcal{C}$.

We define $H^*(\mathcal{C}, \underline{R})$ to be the cohomology groups in the last theorem. This is the cohomology of $\mathcal{C}$.
More generally, for any representation M of $\mathcal{C}$ we put $H^*(\mathcal{C}, M) := \mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, M)$.

Example:

- When $\mathcal{C}$ is a (discrete) group the nerve is the classifying space $B\mathcal{C}$ and the algebraically computed cohomology is isomorphic to the cohomology of $B\mathcal{C}$.

Category extensions: Definition 1

Extension definition EZ:
An extension of a category $\mathcal{C}$ is a diagram of categories and functors

$$\mathcal{K} \to \mathcal{E} \to \mathcal{C}$$

which behaves like a group extension

$$1 \to K \to E \to G \to 1$$

(i.e. a short exact sequence of groups).

Category extensions: Definition 2

An extension of a category $\mathcal{C}$ (in the sense of Hoff) is a diagram of categories and functors

$$\mathcal{K} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{C}$$

satisfying

1. $\mathcal{K}$, $\mathcal{E}$ and $\mathcal{C}$ all have the same objects, i and p are the identity on objects, i is injective on morphisms, and p is surjective on morphisms;
2. whenever f and g are morphisms in $\mathcal{E}$ then $p(f) = p(g)$ if and only if there exists a morphism $m \in \mathcal{K}$ for which $f = i(m)g$. In that case, the morphism m is required to be unique.

Extension properties

Given an extension $\mathcal{K} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{C}$ it follows (not obviously) that

- all morphisms in $\mathcal{K}$ are endomorphisms, and are invertible,
- we get a functor $\mathcal{E} \to$ Groups, $x \mapsto \mathrm{End}_{\mathcal{K}}(x)$.

If all the groups $\mathrm{End}_{\mathcal{K}}(x)$ are abelian

- we get a functor $\mathcal{C} \to$ AbelianGroups

i.e. a representation of C, which we denote K.

Compare: for a group extension $1 \to K \to E \to G \to 1$ there is a conjugation action of E on the normal subgroup K. When K is abelian it becomes a representation of G.

Second cohomology parametrizes extensions

Theorem

When all the groups $\mathrm{End}_{\mathcal{K}}(x)$ *are abelian, equivalence classes of extensions* $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ *biject with elements of* $H^2(\mathcal{C}, K)$.

Other interpretations of cohomology

There are known interpretations of H^1, H^0, H_0, H_1 which generalize to categories the familiar results for groups.
A generalization to categories of the group-theoretic interpretation of H_2 has not previously been observed.

Schur multiplier basics

The Schur multiplier of a category $\mathcal{C}$ is defined to be $H_2(\mathcal{C},\mathbb{Z}) = \mathrm{Tor}_2^{R\mathcal{C}}(\underline{\mathbb{Z}},\underline{\mathbb{Z}})$. This generalizes the definition for groups.

Theorem

Let G be a group for which G/G' is free abelian. There is universal central extension $1 \to K \to E \to G \to 1$ with $K \subseteq E'$, unique up to isomorphism. For that extension, $K \cong H_2(G)$.

central: $K \subseteq Z(E)$
universal: every such extension is a homomorphic image of this one.

Central extension of categories

Questions:

1. What is a central extension $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ of categories?
2. What is the generalization of $K \subseteq E'$ to categories?

Answers:

1. $\mathcal{K}$ is a constant functor. Better: a locally constant functor (=constant on connected components).
2. $H_1(\mathcal{E}, \mathbb{Z}) \to H_1(\mathcal{C}, \mathbb{Z})$ should be an isomorphism.

Universal central extension

Theorem (Webb)

Let $\mathcal{C}$ be a connected category for which $H_1(\mathcal{C})$ is free abelian and $H_2(\mathcal{C})$ is finitely generated. Among extensions $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ where $\mathcal{K}$ is constant and $H_1(\mathcal{E}) \to H_1(\mathcal{C})$ is an isomorphism, there is up to isomorphism a unique one with the property that it has every such extension as a homomorphic image. In this extension $\mathcal{K}$ has the form $\underline{H_2(\mathcal{C})}$.

Methods of proof

- Five-term exact sequences

Theorem (Webb)

Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension of categories, let B be a right $\mathbb{Z}\mathcal{C}$-module and let A a left $\mathbb{Z}\mathcal{C}$-module. There are exact sequences

$$H_2(\mathcal{E}, B) \to H_2(\mathcal{C}, B) \to B \otimes_{\mathbb{Z}\mathcal{C}} H_1(\mathcal{K}) \to H_1(\mathcal{E}, B) \to H_1(\mathcal{C}, B) \to 0$$

and

$$H^2(\mathcal{E}, A) \leftarrow H^2(\mathcal{C}, A) \leftarrow \mathrm{Hom}_{\mathbb{Z}\mathcal{C}}(H_1(\mathcal{K}), A) \leftarrow H^1(\mathcal{E}, A) \leftarrow H^1(\mathcal{C}, A) \leftarrow 0.$$

- Construction of a resolution (Gruenberg resolution) given a surjection $\mathcal{F} \to \mathcal{C}$ where $\mathcal{F}$ is a free category.

The Hopf fibration

Take a category $\mathcal{C}$ whose nerve is a 2-sphere S^2 (for example, take a triangulation of S^2 and let $\mathcal{C}$ be the poset of the simplices).
We have $H^1(\mathcal{C}) = 0$, $H^2(\mathcal{C}) = \mathbb{Z}$, so there is a universal constant extension

$$\underline{\mathbb{Z}} \to \mathcal{E} \to \mathcal{C}$$

Then $|\mathbb{Z}| \to |\mathcal{E}| \to |\mathcal{C}|$ is the Hopf fibration $S^1 \to S^3 \to S^2$.

Theme

Representations of categories are not always like representations of groups!

Finite generation of cohomology

Question: When is the cohomology ring $H^*(\mathcal{C}, \underline{R}) = \mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, \underline{R})$ finitely generated?

Presumably we should put some finiteness conditions on $\mathcal{C}$. Suppose that $\mathcal{C}$ is finite. Also suppose $\mathcal{C}$ is an EI category: every Endomorphism is an Isomorphism (endomorphism monoids are groups).

Evidence for finite generation: it's true when $\mathcal{C}$ is a finite group (Evens-Venkov). When $\mathcal{C}$ is a free category or a poset the cohomology ring is finite dimensional.

Answer (Xu): For a finite EI category the cohomology ring is very often not finitely generated.

Example of non-finite generation of cohomology

Let $\mathcal{C}$ be the category with two objects x, y and seven morphisms as pictured below:

$$C_2 \times C_2 = G \times H \quad \underset{x}{\bullet} \overset{\{\alpha,\beta\}}{\rightrightarrows} \underset{y}{\bullet} \quad 1$$

Here $\mathrm{End}(x) = G \times H$, $\mathrm{End}(y) = 1$ and there are two homomorphisms $\alpha, \beta : x \to y$. Composition is determined by letting G interchange α and β, and letting H fix them.

Proposition (Xu et al)

$H^*(\mathcal{C}, \mathbb{F}_2)$ is isomorphic to the subring of $\mathbb{F}_2[u, v]$ spanned by the monomials $u^r v^s$ where $r \geq 1$.

This ring is not finitely generated and is a domain.

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The conjecture of Snashall and Solberg

Conjecture (Snashall and Solberg, Proc. LMS 88 (2004))

Let A be a finite dimensional algebra over a field. Then the Hochschild cohomology $HH^*(A)$ is finitely generated modulo nilpotent elements.

Here $HH^*(A) := \mathrm{Ext}^*_{A^{\mathrm{op}}\otimes A}(A, A)$.

The conjecture was verified by Green, Snashall and Solberg for self-injective algebras of finite representation type (2003) and 'monomial' algebras (2006) (path algebras of quivers with monomial relations of length 2).

Xu's counterexample

Theorem (Fei Xu, Adv. Math 219 (2008))

Let $k\mathcal{C}$ be the category algebra of a category $\mathcal{C}$ over a field k. The ring homomorphism $HH^*(k\mathcal{C}) \to H^*(\mathcal{C}, k)$ induced by the functor $- \otimes_{k\mathcal{C}} \underline{k}$ is a split surjection.

This result was already known for group algebras. For category algebras it required a new idea.

Corollary

The Snashall-Solberg conjecture is false in general.

For the proof we observe that if $HH^*(A)$ is finitely generated modulo nilpotents, so is every homomorphic image of this ring. Taking $A = k\mathcal{C}$ where $\mathcal{C}$ is the previously described category, we get an image with no nilpotent elements which is not finitely generated.

The use of category representations?

Why did we need to know about representations of categories to do this?

Simple representations of an EI category

If $\mathcal{C}$ is an EI category, the simple representations have the form $S_{x,V}$ where x is an object of $\mathcal{C}$ and V is a simple $k\,\mathrm{End}_{\mathcal{C}}(x)$-module:

$$S_{x,V}(y) = \begin{cases} V & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

This gives a parametrization of the indecomposable projective modules: $P_{x,V}$ is the projective cover of $S_{x,V}$.

The relation

$(x, V) \leq (y, W)$ if and only if there exists a morphism

$x \longrightarrow y$ in $\mathcal{C}$

is a preorder.

Stratifications of algebras

The category algebra $k\mathcal{C}$ is standardly stratified (Cline-Parshall-Scott, Dlab) if there are modules $\Delta_{x,V}$ such that

- all composition factors $S_{y,W}$ of $\Delta_{x,V}$ have $(y, W) \leq (x, V)$, and
- there is a filtration of $P_{y,W}$ with factors $\Delta_{x,V}$ where $(y, W) < (x, V)$, except for a single copy of $\Delta_{y,W}$.

Theorem (Webb (J. Algebra 320 (2008)))

Let $\mathcal{C}$ be a finite EI-category and k a field. Then $k\mathcal{C}$ is standardly stratified if and only if for every morphism $\alpha : x \to y$ in $\mathcal{C}$ the group $\mathrm{Stab}_{\mathrm{Aut}(y)}(\alpha)$ has order invertible in k.

The p-subgroup orbit category

Let G be a finite group and let $\mathcal{O}$ be the category with objects the transitive G-sets G/H where H is a p-subgroup of G. The morphisms are the equivariant mappings of G-sets.

The morphisms are always surjective, and so the criterion for standard stratification is always satisfied, and $\mathcal{O}$ is an EI category.

Corollary

Over any field k the category algebra $k\mathcal{O}$ is standardly stratified.

Further structure

Because $k\mathcal{O}$ is standardly stratified it also has modules

- $\overline{\nabla}_{x,V}$ = largest submodule of the injective $I_{x,V}$ with composition factors smaller than $S_{x,V}$, except for a single copy of $S_{x,V}$
- (partial) tilting modules $T_{x,V}$. They have a filtration with Δ factors, and also a filtration with $\overline{\nabla}$ factors.

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Structural versions of AWC

Theorem

The following are equivalent.

(1) $\Delta_{x,V} = S_{x,V}$ *is a simple* $k\mathcal{O}_{\mathcal{S}}$*-module,*

(2) $\overline{\nabla}_{x,V} = I_{x,V}$ *is injective,*

(3) (x, V) *is a weight:* V *is a projective simple module.*

Theorem

The following are equivalent.

(1) $\Delta_{x,V} = T_{x,V}$,

(2) $\Delta_{H,V} = I_{H,V}$ *is injective,*

(3) $x = G/1$, V *is a simple* kG*-module.*

This gives structural reformulations of Alperin's weight conjecture: the number of weights equals the number of simple kG-modules.

References

Available from http://www.math.umn.edu/ webb

An introduction to the representations and cohomology of categories pp. 149-173 in: M. Geck, D. Testerman and J. Thvenaz (eds.), Group Representation Theory, EPFL Press (Lausanne) 2007.

Resolutions, relation modules and Schur multipliers for categories J. Algebra, to appear.

Standard stratifications of EI categories and Alperin's weight conjecture Journal of Algebra 320 (2008), 4073-4091.

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For more in this direction:

Liping Li: Representation types of finite EI categories, 4:30 today in Combinatorial Representation Theory II, Olin-Rice 241.

An apology ...

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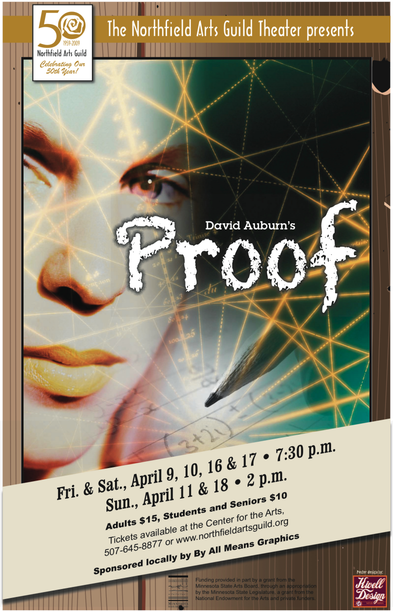

Look at
www.northfieldartsguild.org
for information.
The show is in Northfield,
about 40 miles to the south of
here.
I play the role of Robert.

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