Representations and Cohomology of Categories

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Outline

The most substantial results I will talk about are work of my student, Liping Li. They will appear in his thesis.

A question about cyclic p-groups

Let $G = C_{p^n}$ be a cyclic p-group where p is a prime, $k = \mathbb{F}_p$.
Consider the category of homomorphisms of kG-modules.
Objects: diagrams $M \to N$ of finite dimensional kG-modules.
Morphisms: commutative squares.
We have direct sums of objects:

$$(M \to N) \oplus (M_1 \to N_1) := (M \oplus M_1) \to (N \oplus N_1).$$

An object is indecomposable if it is not (isomorphic to) a non-trivial direct sum.

Question:

How many isomorphism types of indecomposable objects are there?

Are there finitely many? Infinitely many?

The evidence!

We are dealing with $kG = \mathbb{F}_p C_{p^n}$.

There are only finitely indecomposable kG-modules $V_1, \ldots, V_{p^n}$. There are up to isomorphism only finitely many morphisms $V_i \to V_j$ for each i, j. It is hard to see how to combine these morphisms together in many ways.

If $0 \to A \to B \to C \to 0$ is a short exact sequence of kG-modules, it is indecomposable if and only if $B \to C$ is indecomposable. (Exercise) There only seem to be finitely many indecomposable short exact sequences.

Example: for $\mathbb{F}_3 C_3$ there is an indecomposable morphism $(V_3 \oplus V_1) \to (V_3 \oplus V_2)$. ($V_3$ maps to the second layer of V_3 and onto V_2, V_1 maps to the socle of V_2).
Any morphism $(V_1 \oplus V_3 \oplus V_1) \to (V_3 \oplus V_2)$ decomposes.

Would you bet on it?

What do you think?
Are there only finitely many indecomposable morphisms?
Infinitely many indecomposable morphisms?

Would you take me on with a bet?

How do I know that the morphisms in the last example are decomposable or indecomposable?

Representations

Let R be a commutative ring with 1. A representation of a category $\mathcal{C}$ over R is a functor $M : \mathcal{C} \to R$-mod.
A homomorphism of representations is a natural transformation of the functors.

Straightforward example:
$\mathcal{C}$ is the category with five morphisms $\bullet \longleftarrow \bullet \longrightarrow \bullet$.
A representation is a diagram of modules $B \longleftarrow A \longrightarrow C$.
We may be interested in

- the direct limit of this diagram: the pushout;
- is this operation exact?
- Etc.

We DO understand what a representation of a category is!

A representation of a category is a diagram of modules.

Well-studied examples of representations

- When $\mathcal{C}$ is a group we get homomorphism $\mathcal{C} \to \mathrm{End}_R(V)$.
- When $\mathcal{C}$ is a poset we get a module for the incidence algebra.
- When $\mathcal{C}$ is the free category associated to a quiver we get a representation of the quiver.

The theory of representations of the above examples is well developed and we do not expect to get more information about them from this general theory. We are more interested in other categories, such as the orbit category associated to a family of subgroups of a group, or the categories which arise with p-local finite groups.

Further examples

- $\mathcal{C}$ = finite dimensional vector spaces over some field. We get generic representation theory.
- Various constructions in topology and the cohomology of groups: homotopy colimits, the Quillen category.

Back to the question

Let G be a group. Consider the category

$$\mathcal{C} = G \underset{x}{\bullet} \xrightarrow{{}_G G_G} \underset{y}{\bullet} G$$

There are two objects, labelled x and y, with $\mathrm{End}(x) \cong \mathrm{End}(y) \cong G$. The morphisms from x to y are the biset ${}_G G_G$ with composition given by the action of the two groups G on the biset.

A representation M of $\mathcal{C}$ is the specification of RG-modules $M(x)$ and $M(y)$ together with linear maps $M(g) : M(x) \to M(y)$ for each g in ${}_G G_G$.

Since $g1 = 1g$ we have $M(g)M(1) = M(1)M(g)$. Thus $M(1)$ is an RG-module homomorphism, and it determines all the $M(g)$.

Reformulation of the question

A homomorphism of these representations is a commutative diagram of RG-modules.

The 'category of homomorphisms' is the same as the category of representations of

$$G \underset{x}{\bullet} \xrightarrow{{}_G G_G} \underset{y}{\bullet} G.$$

The Category Algebra

The category algebra $R\mathcal{C}$ is the free R-module with the morphisms of $\mathcal{C}$ as a basis. We define the product of these basis elements to be composition if possible, zero otherwise. Examples:

- When $\mathcal{C}$ is a group we get the group algebra.
- When $\mathcal{C}$ is a poset we get the incidence algebra.
- When $\mathcal{C}$ is the free category associated to a quiver we get the path algebra of the quiver.
- When $\mathcal{C} = G \underset{x}{\bullet} \xrightarrow{{}_G G_G} \underset{y}{\bullet} G$ we get

$$\begin{pmatrix} RG & RG \\ 0 & RG \end{pmatrix}$$

Equivalence of representations and modules

Theorem (B. Mitchell)

Representations are 'the same' as $R\mathcal{C}$-modules, if $\mathcal{C}$ has finitely many objects.

Example:

- When $\mathcal{C}$ is a group, representations are the same as modules for the group algebra.
- When $\mathcal{C}$ is the free category associated to a quiver, representations are the same as modules for the path algebra.

Under this correspondence a representation M corresponds to an $R\mathcal{C}$-module $\bigoplus_{x\in \mathrm{Ob}\mathcal{C}} M(x)$. Natural transformations of functors correspond to module homomorphisms.

The question again

What is the representation type of

$$\begin{pmatrix} \mathbb{F}_p C_{p^n} & \mathbb{F}_p C_{p^n} \\ 0 & \mathbb{F}_p C_{p^n} \end{pmatrix}?$$

EI-Categories

This work is done mainly by Liping Li, and the principal results are his.

An EI-category is one in which every Endomorphism is an Isomorphism.

The problem: classify EI categories of finite representation type.

Free EI-Categories

An EI-quiver is a structure $\mathcal{Q} = (Q_0, Q_1, s, t, f, g)$ where (Q_0, Q_1, s, t) is a quiver, $f : Q_0 \to \mathrm{Groups}$, and for each edge $\alpha : x \to y$ in Q_1, $g(\alpha)$ is a $(f(y), f(x))$-biset.

We may construct the free EI-category on $\mathcal{Q}$ as follows. It has objects Q_0 and morphisms $\mathrm{Mor}(x, y) := \coprod g(\alpha_n) \circ \cdots \circ g(\alpha_1)$ where the disjoint union is taken over all chains of arrows $x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} x_n = y$ in Q_1 and $\circ$ indicates the biset product over the common group of the bisets.

A free EI-category is one which is equivalent to a free EI-category on some EI-quiver.

Characterization of free EI-categories.

Free EI-categories are characterized by a universal property and by the uniqueness of factorization of morphisms as words in the generators.

Hereditary category algebras

Theorem (Liping Li, 2010)

Let $\mathcal{C}$ be a free EI-category and let R be a field in which the orders of all groups $\mathrm{End}(x)$ are invertible. Then the category algebra $R\mathcal{C}$ is hereditary.

Li gives an explicit construction of the Ext quiver of $R\mathcal{C}$ and this gives a complete description of the representation type of $R\mathcal{C}$. When $\mathcal{C}$ is an EI category which is not free, it is an image of a free EI-category, and if the free EI-category has finite representation type, so does $\mathcal{C}$.

This result plays a role in the determination of representation type when R is of arbitrary characteristic.

Representation type in positive characteristic

Theorem (Liping Li, 2010)

Let $\mathcal{C}$ be a finite EI-category and R a field of positive characteristic $p \geq 5$. Suppose that $R\mathcal{C}$ has finite representation type. Then

1. *for all objects x of $\mathcal{C}$, $\mathrm{End}(x)$ has cyclic Sylow p-subgroups,*
2. *$O^{p'}\,\mathrm{End}(x)$ acts trivially on $\mathrm{Hom}(x,y)$ and on $\mathrm{Hom}(y,x)$ for all objects $y \neq x$ of $\mathcal{C}$, and*
3. *the quotient category formed by factoring out $O^{p'}(\mathrm{End}(x))$ from $\mathrm{End}(x)$ for each object x has finite representation type.*

Li asks whether the converse is true when $p \geq 5$. Note that in the third condition the quotient category has the property that all automorphism groups have order invertible in R.

Categories with p-group endomorphisms

Theorem (Li, *Webb*)

Let R be a field of positive characteristic p and let $\mathcal{C}$ have two objects x and y. Suppose $\mathrm{End}(x)$ and $\mathrm{End}(y)$ are cyclic p-groups and there is only one other morphism in $\mathcal{C}$. Then $\mathcal{C}$ has finite representation type.

This result says that for an EI-category with two objects whose endomorphism groups are p-groups, the three conditions of the last theorem imply finite representation type.

The answer

Proposition (Li, *Webb*)

When p is a prime and $G = C_{p^n}$ is a cyclic p-group the category

$$G\bullet \xrightarrow{{}_G G_G} \bullet G$$

has finite representation type if and only if $p^n = 2$ or 3.

When $p^n = 2$ there are 8 indecomposables and when $p^n = 3$ there are 27 indecomposables. The method of proof is to calculate the Auslander-Reiten quiver. When $p^n \geq 3$ there is a node in the tree class of the stable quiver of valence 3. We eliminate type D. We show there is a periodic vertex. This produces an upper bound for the number of modules if we have finite type. We construct more than this number if $p^n > 4$.

Some other developments

- The Schur multiplier of a category.
- Fei Xu's counterexample to the conjecture that Hochschild cohomology is finitely generated module nilpotent elements.
- The orbit category and Alperin's weight conjecture.