Combinatorial Restrictions on the AR Quiver of a Triangulated Category

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Outline

A theorem

Additive functions

Proof of the theorem

Another theorem

What might be true?

Definitions and theorem

This is work done with Marju Purin and Kos Diveris.

$\mathcal{C}$ is a $k = \bar{k}$-linear triangulated category which is Hom-finite, connected (or indecomposable) and Krull-Schmidt.

$\mathcal{C}$ is locally finite if for each X there are only finitely many isomorphism classes of objects Y for which $\mathrm{Hom}_{\mathcal{C}}(X, Y) \neq 0$ or $\mathrm{Hom}_{\mathcal{C}}(Y, X) \neq 0$. This condition implies that AR triangles exist and that there is a single AR quiver component.

It is not a hypothesis that all AR triangles exist in the following:

Theorem

Let Γ be a stable component of the AR quiver with Dynkin tree class. Then Γ is the whole of $\mathcal{C}$. Furthermore $\mathcal{C}$ is locally finite. It follows that $\mathcal{C}$ has Auslander-Reiten triangles and has only finitely many shift-classes of indecomposable objects.

Background

1. A stable component Γ is a connected component of the quiver with the property that Auslander triangles exist to the left and right of each object in the component. It has the form $\mathbb{Z}T/G$ where T is a tree (the tree class) and G is a group (Riedtmann, Xiao-Zhu).
2. The theorem is a version for triangulated categories of Auslander's theorem.
3. The theorem was proved by Scherotzke in the case of bounded derived categories of algebras. She related irreducible morphisms in the derived category to irreducible morphisms of ordinary complexes – unlike our method.
4. Locally finite categories were shown to have Dynkin tree class by Xiao and Zhu and (partially) classified by Amiot.

Additive function definitions

A function $\phi : \Gamma_0 \to \mathbb{Z}$ is additive if on each mesh

$$
\begin{array}{ccccc}
 & & E_1 & & \\
 & \nearrow & & \searrow & \\
\tau U & & \vdots & & U \\
 & \searrow & & \nearrow & \\
 & & E_r & &
\end{array}
$$

we have $\phi(U) + \phi(\tau U) = \phi(E_1) + \cdots + \phi(E_r)$.

ϕ is positive if it takes non-negative values, and somewhere is positive.

ϕ is defective on this mesh if
$\phi(U) + \phi(\tau U) = \phi(E_1) + \cdots + \phi(E_r) + 1$.

Additive function lemmas: 1

Lemma

1. *If S is a slice of $\mathbb{Z}T$ and f is an additive function then $f(\tau S) = cf(S)$, where c is a Coxeter transformation.*
2. *If T is a Dynkin tree there is no positive additive function on $\mathbb{Z}T$.*
3. *If T is an extended Dynkin diagram then positive additive functions on $\mathbb{Z}T$ are bounded, and hence periodic.*

Additive function lemmas: 2

If M and X are objects of $\mathcal{C}$ and M is indecomposable put
$f_M(X) := \dim \mathrm{Hom}_{\mathcal{C}}(M, X)$ and
$g_M(X) := \dim \mathrm{Hom}_{\mathcal{C}}(X, M)$.

Lemma

Let Γ be a stable AR quiver component.

1. *f_M is additive on every mesh of Γ except meshes with M and $M[1]$ as the right hand object.*
2. *g_M is additive on every mesh of Γ except meshes with M and $M[-1]$ as the left hand object.*

On the excluded meshes the functions are defective.

Outline of the proof of the theorem

Step 1:

$$\mathcal{C} = \bigcup_{i \in \mathbb{Z}} \Gamma[i].$$

Step 2:

When Γ is finite it is closed under shift.

Step 3:
When Γ is infinite it is closed under shift and locally finite.

Extended Dynkin diagram theorem

Theorem

Let $\mathcal{C}$ be a triangulated category with stable AR quiver components Γ_1 and Γ_2. Suppose there are $X \in \Gamma_1$ and $Y \in \Gamma_2$ so that either $\mathrm{Hom}(X, Y) \neq 0$ or $\mathrm{Hom}(Y, X) \neq 0$. If Γ_1 has an extended Dynkin diagram as its tree class then the tree class of Γ_2 is either an extended Dynkin diagram or one of the trees A_∞, B_∞, C_∞, D_∞ or A_∞^∞.

Proof of the extended Dynkin diagram theorem

Suppose Γ_1 has an extended Dynkin diagram as its tree class and Γ_2 is another component.

If Γ_2 is a shift of Γ_1 then it has the same tree class, which is an extended Dynkin diagram, and we are done. Thus we may suppose that Γ_2 is not a shift of Γ_1. For each M in Γ_2 the functions f_M and g_M are therefore additive on Γ_1, and they are non-negative. It follows that they are always bounded.

Since $\mathrm{Hom}(\tau^n M, X) \cong \mathrm{Hom}(M, \tau^{-n} X)$ it follows that all functions f_X and g_X are bounded on Γ_2 when $X \in \Gamma_1$, and hence periodic. We can find a non-zero such function. This enables us to put a non-negative additive function on the tree of Γ_2, which implies that it must be one of the trees listed.

Questions

Let $\mathcal{C}$ be a Hom-finite, connected, Krull-Schmidt triangulated category.

Let $\mathcal{C}$ have a stable AR quiver component with an extended Dynkin diagram as tree class. Is it necessarily the case that other components have either the same tree class or A_∞?

Is it possible that $\mathcal{C}$ might have two different extended Dynkin diagrams as tree classes of different components?