Canonical Mackey functors

Peter Webb

University of Minnesota

2 August 2016

Outline

What is a Mackey functor?

A Mackey functor M for a group G over a ring R returns an R-module $M(K)$ for each subgroup K of G. It has restriction, induction and conjugation maps.
Conjugation maps $c_g : M(K) \to M(K)$ are supposed to act trivially if $g \in K$, so $M(K)$ is an $RN_G(K)/K$-module.

Examples

- If V is any RG-module the fixed point and fixed quotient functors FP_V and FQ_V are defined by $FP_V(K) := V^K$ and $FQ_V(K) := V_K$.
- For fixed n and V, the cohomology groups $M(K) = H^n(K, V)$ define a Mackey functor. Other examples include various Grothendieck groups of group rings, such as $M(K) = \mathrm{Irr}(K)$.

Axioms

A Mackey functor over R is a mapping $M : \{\text{subgroups of } G\} \to R\text{-mod}$ with morphisms $I_K^H : M(K) \to M(H)$, $R_K^H : M(H) \to M(K)$, $c_g : M(H) \to M({}^gH)$ whenever $K \leq H$ and $g \in G$, such that

- $I_H^H, R_H^H, c_h : M(H) \to M(H)$ are the identity morphisms for all subgroups H and $h \in H$,
- $R_J^K R_K^H = R_J^H$
- $I_K^H I_J^K = I_J^H$ for all subgroups $J \leq K \leq H$,
- $c_g c_h = c_{gh}$ for all $g, h \in G$,
- $R_{{}^gK}^{{}^gH} c_g = c_g R_K^H$
- $I_{{}^gK}^{{}^gH} c_g = c_g I_K^H$ for all subgroups $K \leq H$ and $g \in G$,
- $R_J^H I_K^H = \sum_{x \in [J \backslash H / K]} I_{J \cap {}^xK}^J \, c_x \, R_{J^x \cap K}^K$ for all subgroups $J, K \leq H$.

Representation-theoretic approach to Mackey functors

- The Mackey functors for G are the objects of an abelian category $\mathrm{Mack}_R(G)$.
- These functors are modules for an algebra $\mu_R(G)$ called the Mackey algebra.
- There are projective and injective Mackey functors, simple Mackey functors, blocks of Mackey functors, etc.
- Induction and restriction of Mackey functors between groups $H \leq G$ can be defined using the morphism $\mu_R(H) \to \mu_R(G)$. The are both the left and right adoint of each other, so are exact and preserve projectives and injectives.
- See Thévenaz-Webb, *The structure of Mackey functors,* TAMS 1995.

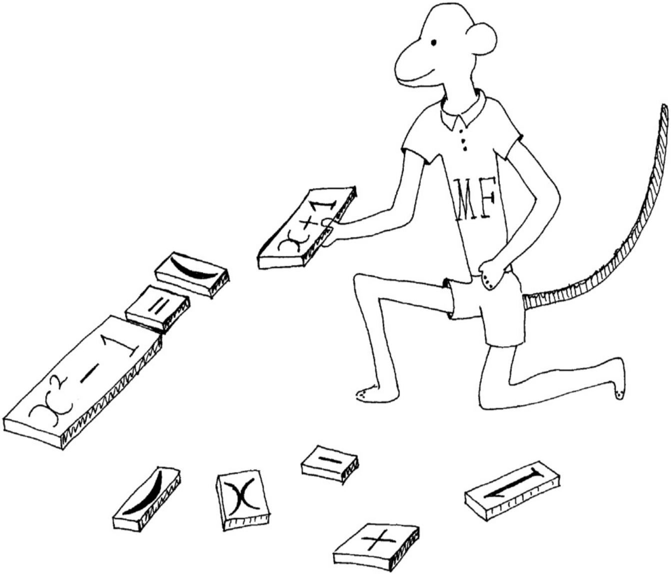
Simple Mackey functors

Theorem (Thévenaz-Webb (1990))

Let $H \le G$ and let V be a a simple $R[N_G(H)/H]$-module. Then $(\mathrm{Inf}_{N(H)/H}^{N(H)} FP_V) \uparrow_{N(H)}^{G}$ has a unique simple subfunctor $S_{H,V}$. These functors $S_{H,V}$ form a complete list of the simple Mackey functors.

The set of pairs H, V that index the simple Mackey functors comes with a natural preorder given by inclusion of the subgroups H.

Did you ever see a monkey factor?



The Δ and ∇ functors: 2001 theory

For any subgroup $H \leq G$ and $RN_G(H)/H$-module U we define
$\Delta_{H,U} = (\mathrm{Inf}_{N(H)/H}^{N(H)} FQ_U) \uparrow_{N(H)}^{G}$ and
$\nabla_{H,U} = (\mathrm{Inf}_{N(H)/H}^{N(H)} FP_U) \uparrow_{N(H)}^{G}$

Proposition

There are formulas:

$$\Delta_{H,U}(K) = \bigoplus_{g \in [K \backslash N_G(H,K)/N_G(H)]} U_{N_{K^g}(H)},$$

and

$$\nabla_{H,U}(K) = \bigoplus_{g \in [K \backslash N_G(H,K)/N_G(H)]} U^{N_{K^g}(H)}.$$

Thus $\Delta_{H,U}(H) = U = \nabla_{H,U}(H)$;
both $\Delta_{H,U}(K)$ and $\nabla_{H,U}(K)$ vanish unless K contains a conjugate of H.

Adjoint characterizations

Proposition

$U \mapsto \nabla_{H,U}$ is right adjoint to taking the Brauer quotient $M \mapsto \overline{M}(H)$.
Similarly $U \mapsto \Delta_{H,U}$ is left adjoint to taking the restriction kernel $\underline{M}(H)$.

The categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$

We will assume from now on that R is a complete p-local ring and U is a p-permutation $RN_G(H)/H$-module.

Let $\mathcal{F}(\Delta)$ be the full subcategory of Mackey functors that have a finite filtration with factors $\Delta_{H,U}$ where $H \leq G$ and U is a p-permutation $RN_G(H)/H$-module.

Similarly, $\mathcal{F}(\nabla)$ is the category with ∇ factors.

Ext vanishing of Δ and ∇

Let R be a complete p-local ring. Let H and K be subgroups of G and let U and W be p-permutation modules for $R[N_G(H)/H]$ and $R[N_G(K)/K]$.

Theorem

- $\mathrm{Ext}^1_{\mu_R(G)}(\Delta_{H,U}, \Delta_{K,W}) = 0$ *unless* $H >_G K$.
- $\mathrm{Ext}^1_{\mu_R(G)}(\Delta_{H,U}, \nabla_{K,W}) = 0$ *unless* $H =_G K$.
- $\mathrm{Ext}^1_{\mu_R(G)}(\nabla_{H,U}, \nabla_{K,W}) = 0$ *unless* $H <_G K$.

Theorem

- $\mathrm{Hom}_{\mu_R(G)}(\Delta_{H,U}, \Delta_{K,W}) = 0$ *unless* $H \geq_G K$.
- $\mathrm{Hom}_{\mu_R(G)}(\Delta_{H,U}, \nabla_{K,W}) = 0$ *unless* $H =_G K$.
- $\mathrm{Hom}_{\mu_R(G)}(\nabla_{H,U}, \nabla_{K,W}) = 0$ *unless* $H \leq_G K$.

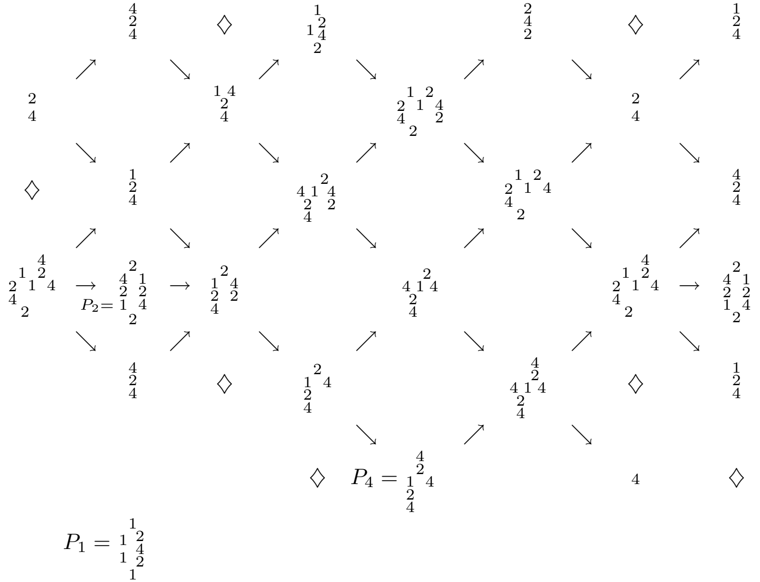
Corollary

$\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ are functorially finite (Ringel). They are closed under direct summands, hence have relative almost split sequences.

Mackey functors in $\mathcal{F}(\Delta)$ for C_4 over $\mathbb{F}_2$

(H,U)	$1, R$	$1, \begin{smallmatrix}R\\R\end{smallmatrix}$	$1, RC_4$	C_2, R	$C_2, \begin{smallmatrix}R\\R\end{smallmatrix}$	C_4, R
Mackey functors $\Delta_{H,U}$	$\begin{smallmatrix}1\\2\\4\end{smallmatrix}$	$\begin{smallmatrix}&1&\\1&&2\\&&4\\&2&\end{smallmatrix}$	$\begin{smallmatrix}&1&\\1&&2\\&&4\\1&&2\\&1&\end{smallmatrix}$	$\begin{smallmatrix}2\\4\end{smallmatrix}$	$\begin{smallmatrix}2\\4\\2\end{smallmatrix}$	4
Ext-injectives $T_{H,U}$	$\begin{smallmatrix}&&&4&\\&&&2&\\4&&1&&4\\&2&&&\\&4&&&\end{smallmatrix}$	$\begin{smallmatrix}&2&\\4&&1\\2&&2\\1&&4\\&2&\end{smallmatrix}$	$\begin{smallmatrix}&1&\\1&&2\\&&4\\1&&2\\&1&\end{smallmatrix}$	$\begin{smallmatrix}4\\2\\4\end{smallmatrix}$	$\begin{smallmatrix}2\\4\\2\end{smallmatrix}$	4
Ext-projectives $\Pi_{H,U}$	$\begin{smallmatrix}1\\2\\4\end{smallmatrix}$	$\begin{smallmatrix}&1&\\1&&2\\&&4\\&2&\end{smallmatrix}$	$\begin{smallmatrix}&1&\\1&&2\\&&4\\1&&2\\&1&\end{smallmatrix}$	$\begin{smallmatrix}&2&\\1&&4\\2&&\\4&&\end{smallmatrix}$	$\begin{smallmatrix}&2&\\4&&1\\2&&2\\1&&4\\&2&\end{smallmatrix}$	$\begin{smallmatrix}&4&\\&2&\\1&&4\\2&&\\4&&\end{smallmatrix}$

Relative AR quiver of $\mathcal{F}(\Delta)$ for C_4 over $\mathbb{F}_2$



Ext projectives and injectives

Theorem

Finitely generated projective Mackey functors lie in $\mathcal{F}(\Delta)$. Finitely generated injective Mackey functors lie in $\mathcal{F}(\nabla)$.

Theorem

The indecomposable Ext-injective and Ext-projective objects in $\mathcal{F}(\Delta)$ are precisely the Ext-injective hulls $I^{\Delta}_{H,U}$ and Ext-projective covers $\Pi^{\Delta}_{H,U}$ of the $\Delta_{H,U}$. In any Δ-filtration, $I^{\Delta}_{H,U}$ always has $\Delta_{H,U}$ at the bottom and $\Pi^{\Delta}_{H,U}$ always has $\Delta_{H,U}$ at the top.

Reformulation of Alperin's weight conjecture

Theorem

- $\Pi_{H,U} = \Delta_{H,U} = I_{H,U} \Leftrightarrow H = 1$ *and* U *is projective.*
  (In this case, $\Pi_{H,U}$ *is projective.)*
- $S_{H,U} = \Delta_{H,U} = I_{H,U} \Leftrightarrow (H, U)$ *is a weight.*

Corollary

Alperin's weight conjecture holds for G *if and only if, among the Mackey functors* $\Delta_{H,U} = I_{H,U}$*, the number that are projective equals the number that are simple.*

New theorem

Theorem (2016)

- $\mathcal{F}(\Delta)^{\perp} \subseteq \mathcal{F}(\nabla)$ *and* ${}^{\perp}\mathcal{F}(\nabla) \subseteq \mathcal{F}(\Delta)$.
- $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \mathcal{F}(\Delta) \cap \mathcal{F}(\Delta)^{\perp} = \mathcal{F}(\nabla) \cap {}^{\perp}\mathcal{F}(\nabla)$.
- *The indecomposable objects in* $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ *are precisely the* Ext*-injective hulls* $I^{\Delta}_{H,V}$ *of the* $\Delta_{H,V}$ *in* $\mathcal{F}(\Delta)$*, and they are precisely the* Ext*-projective covers* $\Pi^{\nabla}_{H,V}$ *of the* $\nabla_{H,V}$ *in* $\mathcal{F}(\nabla)$.
- $I^{\Delta}_{H,V} \cong \Pi^{\nabla}_{H,V}$.
- $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ *is self-dual:* $(I^{\Delta}_{H,V})^{*} \cong \Pi^{\nabla}_{H,V^{*}} \cong I^{\Delta}_{H,V^{*}}$.

Corollary

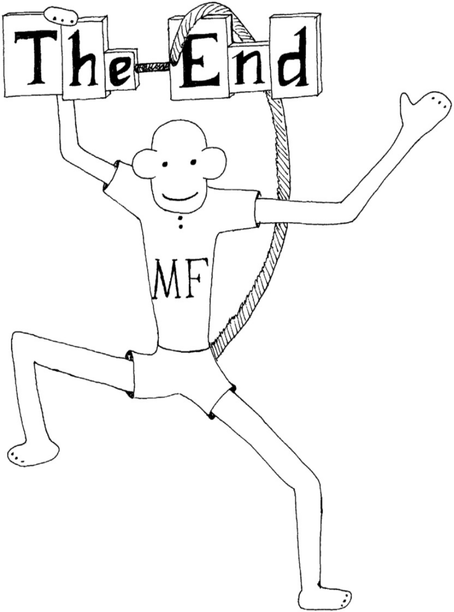
The Ext*-injective hull* $I^{\Delta}_{H,V}$ *of* $\Delta_{H,V}$ *in* $\mathcal{F}(\Delta)$ *also has a* ∇*-filtration. In any* Δ*-filtration of* $I^{\Delta}_{H,V}$ *the bottom term must always be* $\Delta_{H,V}$*. In any* ∇*-filtration of* $I^{\Delta}_{H,V}$ *the top term must always be* $\nabla_{H,V}$*.*

Corollary

Induction and restriction of Mackey functors both preserve $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$*.*

Corollary

If all the p-permutation modules V *are self-dual then the Ringel dual algebra* $\mathrm{End}(\bigoplus I^{\Delta}_{H,V})$ *has symmetric Cartan matrix.*


The End
MF