Globally Defined Mackey Functors and Maps Between Classifying Spaces

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Outline of talk

1. The Deep End
2. Globally defined Mackey functors
3. Highest weight categories
4. Decomposition theory

1. The Deep End

THEOREM. *Let $X_{H,V}$ be a stable summand of $(BH_+)^\wedge_2$ corresponding to the simple $\mathbb{F}_2\operatorname{Out}(H)$-module V, where H is a 2-group. The matrix*

$$\operatorname{rank}_{\mathbb{Z}_p}\operatorname{Hom}(X_{H,V}, X_{K,W})$$

is the Cartan matrix shown, where $\mathcal{X} = \text{all}$ and $\mathcal{Y} = 1$.

THEOREM. *Over any field R, for any class of groups $\mathcal{X}$ closed under extensions and sections, the Cartan matrix $C_R^{\mathcal{X},\mathcal{X}}$ of globally defined Mackey functors $\operatorname{Mack}_R^{\mathcal{X},\mathcal{X}}$ is symmetric and non-singular.*

THEOREM. *Let $(K, \mathcal{O}, k)$ be a p-modular system. Then the Cartan matrices of $\operatorname{Mack}_K^{\mathcal{X},\mathcal{X}}$ and $\operatorname{Mack}_k^{\mathcal{X},\mathcal{X}}$, and decomposition matrix, satisfy $C_k = D^T C_K D$.*

THEOREM. *Over a field R of characteristic zero $\operatorname{Mack}_R^{\mathcal{X},\mathcal{Y}}$ is a highest weight category. $\operatorname{Mack}_R^{1,1}$ is semisimple.*

2. Globally defined Mackey functors

Let $\mathcal{X}$ and $\mathcal{Y}$ be sets of finite groups, closed under taking extensions and sections.

$A_R^{\mathcal{X},\mathcal{Y}}(G,H)$ = Grothendieck group over R of finite (G,H)-bisets with G-stabilizers in $\mathcal{X}$ and H-stablizers in $\mathcal{Y}$.
It is the free R-module with basis the transitive such (G,H)-bisets.

Product: ${}_G\Omega_H \circ {}_H\Psi_K = \Omega \times_H \Psi = \Omega \times \Psi / \sim$
where $(\omega h, \psi) = (\omega, h\psi)$.

Special case: $A_{\mathbb{Z}}^{\text{all},1}(G,G)$ is the *double Burnside ring* which appeared in the 1978 thesis of C. Witten.

The *Burnside category* $\mathcal{C}_R^{\mathcal{X},\mathcal{Y}}$ has
objects = all finite groups in a section-closed class $\mathcal{D}$,
$\text{Hom}(H,G) := A_R^{\mathcal{X},\mathcal{Y}}(G,H)$
with composition of morphisms given by the product.
The special case $\mathcal{C}_R^{\text{all},1}$ appeared in Adams-Gunawardena-Miller, Topology (1985).

A *globally defined Mackey functor* is an R-linear functor $M : \mathcal{C}_R^{\mathcal{X},\mathcal{Y}} \to R\text{-mod}$.
The category of these functors is denoted $\text{Mack}_R^{\mathcal{X},\mathcal{Y}}$.

See Webb, *A guide to Mackey functors,* Handbook of Algebra vol 2, or Webb's site.

Special cases:
when $\mathcal{X}$ = all and $\mathcal{Y} = 1$ these were called 'global Mackey functors' in tom Dieck, Transformation Groups, 1987.
Webb (JPPA 1993) calls these 'inflation functors' and the case $\mathcal{X} = \mathcal{Y} = 1$ 'global Mackey functors'.
Bouc more recently calls these functors 'biset functors'.

The *global Mackey algebra*:
$\mu_R^{\mathcal{X},\mathcal{Y}} = \bigoplus_{G,H} A_R^{\mathcal{X},\mathcal{Y}}(G, H)$.
GDMFs are the same as $\mu_R^{\mathcal{X},\mathcal{Y}}$-modules.

Uses of GDMFs:
1. A method for reducing the computation of $H^*(G)$ to p-groups, by calculating composition factors of $H^*(G)$ as a GDMF (Webb 1993).
2. A proof of the theorems of Nishida, Benson-Feshbach and Martino-Priddy on multiplicities of stable summands of $(BG_+)^\wedge_p$ (Webb 1993).
3. The description of the torsion-free part of the Dade group by Bouc-Thévenaz (2000).

The simple GDMFs $S_{H,V}$ are parametrized by pairs (H, V) where H is a group and V is a simple $R\,\mathrm{Out}(H)$-module (Webb 1993, Bouc 1996).
If R is a field or discrete valuation ring they have projective covers $P_{H,V}$.
The representable functors $\mathrm{Hom}_{\mathcal{C}_R^{\mathcal{X},\mathcal{Y}}}(H,\quad)$ are projective and

$$\mathrm{Hom}_{\mathrm{Mack}_R^{\mathcal{X},\mathcal{Y}}}(\mathrm{Hom}_{\mathcal{C}_R^{\mathcal{X},\mathcal{Y}}}(H,\quad), \mathrm{Hom}_{\mathcal{C}_R^{\mathcal{X},\mathcal{Y}}}(G,\quad)) \cong \mathrm{Hom}_{\mathcal{C}_R^{\mathcal{X},\mathcal{Y}}}(G, H).$$

Carlsson's theorem (the Segal conjecture) implies when H is a p-group that

$$\mathrm{Hom}_{\mathcal{C}_{\mathbb{Z}_p}^{\mathrm{all},1}}(G, H) = A_{\mathbb{Z}_p}^{\mathrm{all},1}(H, G) \cong \mathrm{Hom}((BH_+)_p^\wedge, (BG_+)_p^\wedge).$$

There is an equivalence of full subcategories with the following objects:
summands $P_{H,V}$ of $\mathrm{Hom}_{\mathcal{C}_R^{\mathcal{X},\mathcal{Y}}}(H,\quad)$, H a p-group
summands $X_{H,V}$ of $(BH_+)_p^\wedge$, H a p-group.

Under this equivalence,
$\mathrm{Hom}(P_{H,V}, P_{K,W}) \cong \mathrm{Hom}(X_{H,V}, X_{K,W})$,
the entries in the Cartan matrix up to dimensions of endomorphisms of simples.