Doing group representations with categories

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Outline

1. Main Theorem on the Schur multiplier
2. Basics about representations of categories
3. Stem extensions of categories
4. Example
5. The Gruenberg resolution, relation modules
6. 5-term exact sequences

1. Main Theorem

This work appears in
P.J. Webb, Resolutions, relation modules and Schur multipliers for categories, preprint from my web site.

Assume:

$\mathcal{C}$ is a connected category.

$H_2(\mathcal{C})$ is finitely generated.

THEOREM. *The maximal stem extensions of $\mathcal{C}$ by constant functors are the stem extensions by the constant functor $\underline{H_2(\mathcal{C})}$ with group $H_2(\mathcal{C})$. When $H_1(\mathcal{C})$ is free abelian, they are all isomorphic.*

THEOREM. *Every group extension $1 \to K \to E \to G \to 1$ with $K \leq Z(E) \cap E'$ is an image by extension pushout of such an extension with $K = H_2(G)$. When G/G' is free abelian, these maximal extensions are all isomorphic.*

Linckelmann has defined the Schur multiplier of a fusion system. It is the Schur multiplier of the linking system.

2. Basics about representations of categories

See:

P.J. Webb, An introduction to the representations and cohomology of categories, pp. 149-173 in: M. Geck, D. Testerman and J. Thévenaz (eds.), Group Representation Theory, EPFL Press (Lausanne) 2007.

Dictionary:

group G	category $\mathcal{C}$
representation = homomorphism	representation = functor
group algebra RG	category algebra $R\mathcal{C}$
Examples: group algebra, incidence algebra, path algebra	
representation = module	representation = module
internal $\otimes$	internal $\otimes$
trivial representation R	constant functor $\underline{R}$
fixed points $M^G = \mathrm{Hom}_{RG}(R, M)$	inverse limit $\varprojlim_{\mathcal{C}} M = \mathrm{Hom}_{R\mathcal{C}}(\underline{R}, M)$
cofixed points $M_G = R \otimes_{RG} M$	direct limit $\varinjlim_{\mathcal{C}} M = \underline{R} \otimes_{R\mathcal{C}} M$
classifying space BG	nerve $\lvert\mathcal{C}\rvert$
$H^n(G, M) = \mathrm{Ext}^n_{RG}(R, M)$	$H^n(\mathcal{C}, M) = \mathrm{Ext}^n_{R\mathcal{C}}(\underline{R}, M)$
$H^n(G, \mathbb{Z}) = H^n(BG)$	$H^n(\mathcal{C}, \mathbb{Z}) = H^n(\lvert\mathcal{C}\rvert)$
H^2 corresponds to equivalence classes of extensions.	
split extension $M \rtimes G$	Grothendieck construction $M \rtimes \mathcal{C}$
H^1 corresponds to conjugacy classes of splittings of $M \rtimes \mathcal{C}$.	
augmentation ideal IG	left and right augmentation ideals ${}^\bullet I\mathcal{C}$ and $I\mathcal{C}^\bullet$
$G/G' \cong IG/(IG)^2$	$H_1(\mathcal{C}) \cong \frac{I\mathcal{C}^\bullet \cap {}^\bullet I\mathcal{C}}{I\mathcal{C}^\bullet \cdot {}^\bullet I\mathcal{C}}$
role of Schur multiplier $H_2(G)$	?
central group extension	extension by a (sub) (locally) constant functor
essential extension	?
presentation, relation module	?,?
G-sets	$\mathcal{C}$-sets
bisets for groups	bisets for categories

Applications: p-local finite groups, computing higher limits (Bousfield-Kan spectral sequence); related topological constructions; reformulation of Alperin's conjecture; Bredon coefficient systems; Xu's example; the theory includes representations of quivers and of posets.

3. Stem extensions of categories

Extension of category:

An extension of categories is a pair of functors

$$\mathcal{K}\xrightarrow{i}\mathcal{E}\xrightarrow{p}\mathcal{C}$$

between categories $\mathcal{K}$, $\mathcal{E}$ and $\mathcal{C}$ for which

(1) $\mathcal{K}$, $\mathcal{E}$ and $\mathcal{C}$ all have the same objects, i and p are the identity on objects, i is injective on morphisms, and p is surjective on morphisms;

(2) whenever f and g are morphisms in $\mathcal{E}$ then $p(f) = p(g)$ if and only if there exists a morphism $m \in \mathcal{K}$ for which $f = i(m)g$. In that case, the morphism m is required to be unique.

It follows that $\mathcal{K}$ is a disjoint union of groups, and that we have a functor $K : \mathcal{E} \to$ Groups with $K(x) = \mathrm{End}_{\mathcal{K}}(x)$. When these groups are abelian we get a representation of $\mathcal{E}$, and in fact a representation of $\mathcal{C}$.

It is convenient to use the notation $(\mathcal{K}|\mathcal{E})$ to denote the above extension, although this notation does not retain complete information about it.

Morphism and isomorphism of extensions of $\mathcal{C}$: a commutative diagram which is the identity on $\mathcal{C}$.

Extensions by contant functors generalize central group extensions, as do extensions by locally constant and sublocally constant functors.

Stem extension: an extension $(K|\mathcal{E})$ such that the induced map $H_1(\mathcal{E}) \to H_1(\mathcal{C})$ is an isomorphism.

PROPOSITION. *In any map of extensions*

$$\begin{array}{ccccc} K_1 & \to & \mathcal{E}_1 & \to & \mathcal{C} \\ \downarrow & & \downarrow & & \| \\ K & \to & \mathcal{E} & \to & \mathcal{C} \end{array}$$

where K_1 and K are locally constant and the lower extension is a stem extension, the homomorphism $K_1 \to K$ is surjective.

Proof. Use the 5-term sequence in homology. □

Maximal stem extension: There is a transitive relation $\geq$ on the set of stem extensions of $\mathcal{C}$ by locally constant functors by writing $(K_1|\mathcal{E}_1) \geq (K|\mathcal{E})$ if there is a morphism $(K_1|\mathcal{E}_1) \to (K|\mathcal{E})$. Provided $H_2(\mathcal{C})$ is a finitely generated abelian group, if $(K_1|\mathcal{E}_1) \geq (K|\mathcal{E})$ and $(K|\mathcal{E}) \geq (K_1|\mathcal{E}_1)$ then $(K|\mathcal{E}) \cong (K_1|\mathcal{E}_1)$ since the two composites of the two morphisms when restriced to K and K_1 are both surjections, and hence isomorphisms. Thus $\geq$ induces a partial order on the set of isomorphism classes of stem extensions. We say that a stem extension $(K|\mathcal{E})$ of $\mathcal{C}$ by a locally constant functor is *maximal* if its isomorphism class is maximal in this partial order. Equivalently, a stem extension by a locally constant functor is maximal if and only if whenever it is the target of a morphism from a stem extension, that morphism is an isomorphism.

4. Example: the suspension of $K(C_2, 1)$

Let $\mathcal{C}$ be the category with three objects labelled x, y and z and three non-identity morphisms: $g : x \to x$, $a : x \to y$ and $b : x \to z$ and so that $g^2 = 1_x$, the other compositions being determined uniquely. The nerve of $\mathcal{C}$ is the suspension of the Eilenberg-MacLane space $K(C_2, 1)$, since the endomorphism monoid of x has as its nerve $K(C_2, 1)$, adjoining just one of the morphisms a or b produces a cone on this space, and adjoining both a and b gives the double cone, or suspension. Thus the homology $H_n(\mathcal{C})$ of $\mathcal{C}$ is $\mathbb{Z}$ if $n = 0$, 0 if n is odd and $\mathbb{Z}/2\mathbb{Z}$ if n is even.

We take the presentation of $\mathcal{C}$ by the free category $\mathcal{F}$ with objects x, y and z and generator morphisms $G : x \to x$, $A : x \to y$ and $B : x \to z$. The left augmentation ideal of $\mathcal{F}$ is

$$^{\bullet}I\mathcal{F} = \mathbb{Z}\mathcal{F}(1_x - G) + \mathbb{Z}\mathcal{F}(1_y - A) + \mathbb{Z}\mathcal{F}(1_z - B)$$

and the kernel N of the algebra homomorphism $\mathbb{Z}\mathcal{F} \to \mathbb{Z}\mathcal{C}$ is the 2-sided ideal generated by $1_x - G^2$, $B(1_x - G)$ and $A(1_x - G)$. We find that

$$N \cdot {}^{\bullet}I\mathcal{F} = \mathbb{Z}\mathcal{F}(1_x - G^2)(1_x - G) + \mathbb{Z}A(1_x - G)^2 + \mathbb{Z}B(1_x - G)^2$$

noting that many products of terms vanish in this computation. Making use of the identity $1_x - G^2 = -(1_x - G)^2 + 2(1_x - G)$ we find that, modulo $N \cdot {}^{\bullet}I\mathcal{F}$, N is spanned by $1_x - G^2$, $A(1_x - G)$ and $B(1_x - G)$. The images of these elements in $N/(N \cdot {}^{\bullet}I\mathcal{F})$ are independent (on considering the effects of left multiplication by 1_x, 1_y and 1_z) and we have $A(1 - G^2) \equiv 2A(1 - G)$ and $B(1 - G^2) \equiv 2B(1 - G)$. Thus the relation module M associated to this presentation has $M(x) = M(y) = M(z) = \mathbb{Z}$ and $M(G)$ acts as the identity on $M(x)$, $M(A)$ includes $M(x)$ into $M(y)$ as $\mathbb{Z} \to 2\mathbb{Z}$ and also $M(B)$ includes $M(x)$ into $M(z)$ as $\mathbb{Z} \to 2\mathbb{Z}$.

We see from this that the projective extension of $\mathcal{C}$ described in Theorem ? has as its middle term the category $\mathcal{F}^{\dagger\dagger}$ which has objects x, y and z and with each of the morphism sets $\mathrm{End}(x)$, $\mathrm{End}(y)$, $\mathrm{End}(z)$, $\mathrm{Hom}(x, y)$ and $\mathrm{Hom}(x, z)$ a copy of $\mathbb{Z}$. The composition of any two composable morphisms is the sum of the integers. The surjection $\mathcal{F}^{\dagger\dagger} \to \mathcal{C}$ sends each of these sets of morphisms to a single morphism, except for $\mathrm{End}_{\mathcal{F}}(x) \to \mathrm{End}_{\mathcal{C}}(x)$ which is a surjective group homomorphism $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. This surjection of categories is seen to be part of an extension

$$M \to \mathcal{F}^{\dagger\dagger} \to \mathcal{C}.$$

Since $H_1(\mathcal{C}) = 0$ and $H_2(\mathcal{C}) = \mathbb{Z}/2\mathbb{Z}$ there is a unique (up to isomorphism) maximal stem extension of $\mathcal{C}$ by a locally constant functor, and its left term is the constant functor $\underline{\mathbb{Z}/2\mathbb{Z}}$. We describe it first and then explain how it may be calculated. The extension category $\mathcal{E}$ has objects x, y and z and now $\mathrm{End}(y)$ and $\mathrm{End}(z)$ are copies of the cyclic group C_2 while $\mathrm{End}(x)$ is a Klein four-group $C_2 \times C_2$. We may identity $\mathrm{Hom}(x, y)$ as a copy of C_2 in which $\mathrm{End}(x)$ acts via composition as projection onto the first factor $p_1 : C_2 \times C_2 \to C_2$ and then multiplication within C_2, and $\mathrm{End}(y)$ acts as multiplication within C_2. On the other hand $\mathrm{Hom}(x, z)$ is a copy of C_2 in which $\mathrm{End}(x)$ acts via composition as projection onto the second factor $p_2 : C_2 \times C_2 \to C_2$ and then multiplication within C_2, and $\mathrm{End}(z)$ acts as multiplication within C_2. There is an extension

$$\underline{\mathbb{Z}/2\mathbb{Z}} \to \mathcal{E} \to \mathcal{C}$$

in which the surjection $\mathrm{End}_{\mathcal{E}}(x) \to \mathrm{End}_{\mathcal{C}}(x)$ is the homomorphism $C_2 \times C_2 \to C_2$ which has the diagonally embedded copy of C_2 as its kernel, and this is the maximal stem extension.

We may compute this extension by first working over the field of two elements $\mathbb{F}_2$ and constructing a projective resolution of the constant functor $\underline{\mathbb{F}_2}$ by projective $\mathbb{F}_2\mathcal{C}$-modules. We write modules diagrammatically, so that the constant functor $\underline{\mathbb{F}_2}$ has the form

$$\underline{\mathbb{F}_2} = \begin{matrix} & S_x & \\ S_y & & S_z \end{matrix}$$

where we denote the three simple $\mathbb{F}_2\mathcal{C}$-modules, each of dimension 1 on the subscript object and 0 on the other two, by S_x, S_y and S_z. we have a decomposition of the regular representation into indecomposable projective modules

$$\mathbb{F}_2\mathcal{C} = P_x \oplus P_y \oplus P_z = \begin{matrix} & S_x & \\ S_x \quad & S_y & \quad S_z \end{matrix} \oplus S_y \oplus S_z$$

and the start of a resolution

$$S_x \oplus S_y \oplus S_z \to P_x \to P_x \to \begin{matrix} & S_x & \\ S_y & & S_z \end{matrix} \to 0.$$

Using this we may compute $H^2(\mathcal{C}, \underline{\mathbb{F}_2})$ by the exact sequence

$$\mathrm{Hom}(P_x, \underline{\mathbb{F}_2}) \to \mathrm{Hom}(S_x \oplus S_y \oplus S_z, \underline{\mathbb{F}_2}) \to H^2(\mathcal{C}, \underline{\mathbb{F}_2}) \to 0$$

and we see that $H^2(\mathcal{C}, \underline{\mathbb{F}_2})$ has order 2 with a non-zero element in cohomology represented by the morphism $S_x \oplus S_y \oplus S_z \to \begin{smallmatrix} & S_x & \\ S_y & & S_z \end{smallmatrix} \to 0$ which is non-zero on the S_y summand and zero on the other two summands. Translating this to the relation module which we have already constructed, the stem extension we require is the explicit pushout of the projective extension $M \to \mathcal{F}^{\dagger\dagger} \to \mathcal{C}$ along the morphism $M \to \underline{\mathbb{F}_2}$ given as surjection $\mathbb{Z} \to \mathbb{F}_2$ at the object y and zero at the other objects. Following the definition of the explicit pushout now gives the desired stem extension.

5. Translating between extension of categories and of the left augmentation ideal

The left and right augmentation ideals: the constant functor $\underline{\mathbb{Z}}$ as a $\mathbb{Z}\mathcal{C}$-module is $\bigoplus_{x\in \mathrm{Ob}\mathcal{C}} \mathbb{Z}$. There is a map of left $\mathbb{Z}\mathcal{C}$-modules $\mathbb{Z}\mathcal{C} \to \underline{\mathbb{Z}}$ specified by $\alpha \mapsto 1_{\mathrm{cod}\alpha}$. Its kernel is ${}^{\bullet}I\mathcal{C}$, the span in $\mathbb{Z}\mathcal{C}$ of elements $\alpha - 1_{\mathrm{cod}\alpha}$.

THEOREM. *Let $\mathcal{E}$ and $\mathcal{C}$ be categories with the same objects, let $p : \mathcal{E} \to \mathcal{C}$ be a surjection of categories which is the identity on objects and let $N = \mathrm{Ker}(R\mathcal{E} \to \mathbb{Z}\mathcal{C})$. Consider the acyclic complex of $\mathbb{Z}\mathcal{C}$-modules*

$$\cdots \to \frac{N^t}{N^{t+1}} \to \frac{N^{t-1}\cdot {}^{\bullet}I\mathcal{E}}{N^t \cdot {}^{\bullet}I\mathcal{E}} \to \frac{N^{t-1}}{N^t} \to \cdots$$
$$\to \frac{N}{N^2} \to \frac{{}^{\bullet}I\mathcal{E}}{N\cdot {}^{\bullet}I\mathcal{E}} \to \mathbb{Z}\mathcal{C} \to \underline{R} \to 0.$$

If $\mathcal{E}$ is a free category this complex is an $\mathbb{Z}\mathcal{C}$-projective resolution of $\underline{R}$.

THEOREM. *Let $\mathcal{F}$ and $\mathcal{C}$ be categories with the same objects, let $p : \mathcal{F} \to \mathcal{C}$ be a surjection of categories which is the identity on objects, and let $N = \mathrm{Ker}(\mathbb{Z}\mathcal{F} \to \mathbb{Z}\mathcal{C})$. Suppose that $\mathcal{F}$ is a free category. Then*

$$H_{2n}(\mathcal{C}, \underline{R}) \cong \frac{N^n \cap I\mathcal{F}^{\bullet} \cdot N^{n-1} \cdot {}^{\bullet}I\mathcal{F}}{I\mathcal{F}^{\bullet}\cdot N^n + N^n \cdot {}^{\bullet}I\mathcal{F}}$$

and

$$H_{2n+1}(\mathcal{C}, \underline{R}) \cong \frac{I\mathcal{F}^{\bullet}\cdot N^n \cap N^n \cdot {}^{\bullet}I\mathcal{F}}{N^{n+1} + I\mathcal{F}^{\bullet}\cdot N^n \cdot {}^{\bullet}I\mathcal{F}}.$$

The formula for first homology simplifies to give

$$H_1(\mathcal{C}, \underline{R}) \cong \frac{I\mathcal{C}^{\bullet} \cap {}^{\bullet}I\mathcal{C}}{I\mathcal{C}^{\bullet}\cdot {}^{\bullet}I\mathcal{C}}.$$

This generalizes the formula for groups $G/G' \cong IG/(IG)^2$.

THEOREM. *There is an equivalence of categories between extensions of $\mathcal{C}$ with abelian kernel and extensions of ${}^{\bullet}I\mathcal{C}$, under which $K \to \mathcal{E} \to \mathcal{C}$ corresponds to*

$$0 \to K \to \frac{{}^{\bullet}I\mathcal{E}}{N\cdot {}^{\bullet}I\mathcal{E}} \to {}^{\bullet}I\mathcal{C} \to 0.$$

We have $K \cong \frac{N}{N\cdot {}^{\bullet}I\mathcal{E}}$.

The relation module

The relation module in this situation is $N/(N \cdot {}^{\bullet}I\mathcal{F})$. It appears in a short exact sequence $0 \to N/(N \cdot {}^{\bullet}I\mathcal{F}) \to {}^{\bullet}I\mathcal{F}/(N \cdot {}^{\bullet}I\mathcal{F}) \to {}^{\bullet}I\mathcal{C} \to 0$.

THEOREM. *When $\mathcal{C}$ is a group, the relation module coming from a monoid presentation of $\mathcal{C}$ is isomorphic to the relation module coming from the corresponding group presentation of $\mathcal{C}$. It is also the kernel in a category extension of $\mathcal{C}$ which is projective in a certain category of extensions.*

Application: geometric group theory.

Keywords to explain:

relation module coming from a monoid presentation. In fact there is a relation module coming from every free presentation of a category.

6. Five-term sequences

THEOREM. *(5-term sequences in the (co)homology of a category extension.) Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension of categories, let B be a right $\mathbb{Z}\mathcal{C}$-module and let A a left $\mathbb{Z}\mathcal{C}$-module. There are exact sequences*

$$\begin{aligned} &H_2(\mathcal{E}, B) \to H_2(\mathcal{C}, B) \to \\ &\qquad B \otimes_{\mathbb{Z}\mathcal{C}} H_1(\mathcal{K}) \to H_1(\mathcal{E}, B) \to H_1(\mathcal{C}, B) \to 0 \end{aligned}$$

and

$$\begin{aligned} &H^2(\mathcal{E}, A) \leftarrow H^2(\mathcal{C}, A) \leftarrow \\ &\qquad \mathrm{Hom}_{\mathbb{Z}\mathcal{C}}(H_1(\mathcal{K}), A) \leftarrow H^1(\mathcal{E}, A) \leftarrow H^1(\mathcal{C}, A) \leftarrow 0. \end{aligned}$$

COROLLARY. *Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension of categories and suppose that the induced homomorphism $H_1(\mathcal{E}) \to H_1(\mathcal{C})$ is an isomorphism. Then $\varinjlim H_1(\mathcal{K}) = \underline{\mathbb{Z}} \otimes_{\mathbb{Z}\mathcal{C}} H_1(\mathcal{K})$ is a homomorphic image of $H_2(\mathcal{C})$.*

Techniques:

The equivalence of extension categories of $\mathcal{C}$ and of the left augmentation ideal ${}^{\bullet}\mathcal{C}$. Show that given a stem extension there is indeed a stem extension with kernel group $H_2(\mathcal{C})$ mapping to the given stem extension.