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Stratification of globally defined Mackey functors

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Outline

Notation

- A section of a group is a quotient of a subgroup.
- $\mathcal{X}$, $\mathcal{Y}$ are sets of groups closed under sections and extensions.
- $\mathcal{D}$ is a set of groups closed under sections.
- $A^{\mathcal{X},\mathcal{Y}}(G,H) :=$ Grothendieck group of (G,H)-bisets whose left stabilizers lie in $\mathcal{X}$ and right stabilizers lie in $\mathcal{Y}$.
- $A_R^{\mathcal{X},\mathcal{Y}}(G,H) := R \otimes_{\mathbb{Z}} A^{\mathcal{X},\mathcal{Y}}(G,H)$
- $\mathcal{C}_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D})$ is the category with objects the groups in $\mathcal{D}$ and $\mathrm{Hom}_{\mathcal{C}}(H,G) := A_R^{\mathcal{X},\mathcal{Y}}(G,H)$
- $\mathrm{Mack}_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D}) = \mathrm{Mack}$ is the category of R-linear functors $\mathcal{C}_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D}) \to R$-mod.
  I have been calling the objects of Mack in this generality globally defined Mackey functors.

Some history

1981 Haynes Miller wrote to Frank Adams, describing the Burnside category $:= \mathcal{C}_{\mathbb{Z}}^{\text{all},1}(\text{all})$.

1985 The Burnside category appears in print: Adams, Gunawardena, Miller (Topology 24).

1987 Use of the term global Mackey functor for the objects of $\text{Mack}^{\text{all},1}$, in tom Dieck: 'Transformation Groups', page 278. These functors are defined on a version of $\mathcal{C}_{\mathbb{Z}}^{\text{all},1}$ which is set up to work for compact Lie groups.

1991 Symonds called objects of $\text{Mack}^{\text{all},1}$ functors with Mackey structure in Comment. Math. Helv. 66.

Some history (continued)

1993 In J. Pure Appl. Algebra 88 I used the terms global Mackey functor for objects of $\mathrm{Mack}^{1,1}$ and inflation functors for $\mathrm{Mack}^{\mathrm{all},1}$. These functors were defined on $\mathcal{C}_R^{1,1}$ and $\mathcal{C}_R^{\mathrm{all},1}$. I learnt this approach from topologists, some of whom were calling the objects of $\mathrm{Mack}^{\mathrm{all},1}$ Burnside functors. The algebra over which these functors are modules appears in my paper and was called the global Mackey algebra.

1996 In J. Algebra 183 Bouc considers $\mathcal{C}_R^{\mathcal{X},\mathcal{Y}}$ and $\mathrm{Mack}_R^{\mathcal{X},\mathcal{Y}}$ for arbitrary $\mathcal{X}$ and $\mathcal{Y}$ (not using exactly this notation) and classifies simple functors and many other things in this general context.

2000 In 'A guide to Mackey functors' (Handbook of Algebra vol. 2) I use the term globally defined Mackey functors for objects of $\mathrm{Mack}_R^{\mathcal{X},\mathcal{Y}}$ in general.

200x The term biset functors is used, as well as the alchemic algebra.

Previous talks

- Oberwolfach Dec. 1995: ‘Mackey functors and highest weight categories’
- Seattle July 1996
- Fields Institute August 2002.
  A printed version appeared in Fields Institute Communications 40 (2004), 277-289.

Parametrization of simple globally defined Mackey functors

Theorem

Simple GDMF biject with pairs (H, U) where H is a finite group and U is a simple $R[\mathrm{Out}\, H]$-module (both taken up to isomorphism).
Provided R is a field or a complete discrete valuation ring, each simple functor $S_{H,U}$ has a projective cover $P_{H,U}$, and these form a complete list of the indecomposable projective functors.

The corresponding simple functor $S_{H,U}$ has the property that $S_{H,U}(H) \cong U$ as $R[\mathrm{Out}\, H]$-modules, and that if G is a group for which $S_{H,U}(G) \neq 0$ then H is a section of G.
The classification is independent of the choice of $\mathcal{X}$ and $\mathcal{Y}$, although the particular structure of the simple functors changes as we vary $\mathcal{X}$ and $\mathcal{Y}$.

The global Mackey algebra

The global Mackey algebra is

$$\mu_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D}) = \bigoplus_{G,H\in\mathcal{D}} A_R^{\mathcal{X},\mathcal{Y}}(G,H)$$

as an R-module. The multiplication of two elements which lie in these summands is defined to be composition (= biset multiplication) if the elements can be composed, and zero otherwise.
Globally defined Mackey functors are 'the same thing' as $\mu_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D})$-modules, at least when $\mathcal{D}$ is finite.

GDMFs as μ-modules

A globally defined Mackey functor M may be regarded as a $\mu_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D})$-module

$$\bigoplus_{H\in\mathcal{D}} M(H),$$

with the action of a biset ${}_G\Omega_H$ on the summand $M(H)$ being given by $M({}_G\Omega_H)$, and zero on the other summands.

Conversely, for each group G in $\mathcal{D}$ there is an idempotent

$${}_GG_G \in A_R^{\mathcal{X},\mathcal{Y}}(G,G) \subseteq \mu_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D}),$$

and each $\mu_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D})$-module U gives rise to a globally defined Mackey functor M where $M(G) = {}_GG_G \cdot U$.

This procedure produces an equivalence of categories between $\mathrm{Mack}_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D})$ and $\mu_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D})$-modules U with the property that $U = \bigoplus_{G\in\mathcal{D}} {}_GG_G \cdot U$.

Highest weight categories

The simple GDMFs are parametrized by

$$\Lambda = \{(H, V) \,|\, H \text{ is a group, } V \text{ is a simple } R\,\mathrm{Out}\,H\text{-module, both taken up to isomorphism}\}.$$

We put a partial order on Λ: $(H, V) > (K, W)$ if and only if H is a proper section of K.
Let $\mathcal{O}$ be the full subcategory of $\mathrm{Mack}_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D})$ whose objects are the globally defined Mackey functors defined on $\mathcal{D}$ which are finite dimensional at each evaluation and which are the union of their subobjects of finite length.

Theorem

If R is a field of characteristic zero then $\mathcal{O}$ is a highest weight category with the specified partial order on Λ.

Semisimplicity of GDMF

Theorem

Let R be a field of characteristic zero. Then $Mack_R^{\mathcal{X},\mathcal{Y}}(all)$ is semisimple if and only if $\mathcal{X} = \mathcal{Y} = 1$.

Symmetry and non-singularity of the Cartan matrix

Theorem

Let R be an algebraically closed field, suppose that $\mathcal{X} = \mathcal{Y}$ and let $\mathcal{D}$ be a section-closed set of groups. Then the Cartan matrix of globally defined Mackey functors defined on $\mathcal{D}$ is symmetric.

Theorem

Let R be a field, let $\mathcal{D}$ be a section-closed set of groups and in case R has positive characteristic suppose that $\mathcal{X} = \mathcal{Y}$. Then the Cartan matrix of globally defined Mackey functors defined on $\mathcal{D}$ is non-singular.

Cartan matrix for $\mathrm{Mack}_{\mathbb{F}_2}^{1,1}$ in characteristic 2

Cartanmatrix		$\mathrm{P}_{H,V}$								
$\mathcal{X} = 1, \mathcal{Y} = 1$		1	C_2	C_4	C_2^2		C_8	Q_8		D_8
characteristic 2		1	1	1	1	2	1	1	2	1
$S_{K,W}$	1 1	1	1	1	1		1	1		1
	C_2 1	1	2	2	2	1	2	2		2
	C_4 1	1	2	4	2	1	4	3	1	3
	C_2^2 1	1	2	2	4	1	2	2		4
	2		1	1	1	2	1	1		1
	C_8 1	1	2	4	2	1	8	3	1	3
	Q_8 1	1	2	3	2	1	3	5	1	3
	2			1			1	1	2	1
	D_8 1	1	2	3	4	1	3	3	1	9

Generalization of $C = D^T D$

Theorem

Let $(K, \mathcal{O}, k)$ be a p-modular system, let A_0 be an $\mathcal{O}$-order in a finite dimensional K-algebra A and let D be the decomposition matrix from characteristic zero to characteristic p. Then the Cartan matrix for $k \otimes_{\mathcal{O}} A_0$ is $C_k = D^T C_K D$ where C_K is the Cartan matrix of A.

Brauer quotient and restriction kernel

The Brauer quotient of the GDMF M at the group H is

$$\overline{M}(H) = M(H) / \sum_{\substack{\gamma : J \to H \\ H \text{ not a section of } J}} \operatorname{Im} M(\gamma).$$

The restriction kernel of M at H is

$$\underline{M}(H) = \bigcap_{\substack{\gamma : H \to J \\ H \text{ not a section of } J}} \operatorname{Ker} M(\gamma).$$

When R is a local ring it is usual to factor out in addition the radical of the ring from these expressions, which we do not do here.

Standard and costandard objects

Proposition

For each subgroup H the functor

$$Mack_R^{\mathcal{X},\mathcal{Y}} \to R\,Out\,H\text{-mod}$$

specified by $M \mapsto \underline{M}(H)$ has left adjoint $V \mapsto \Delta_{H,V}$. Similarly the functor specified by $M \mapsto \overline{M}(H)$ has right adjoint $V \mapsto \nabla_{H,V}$. The GDMFs $\Delta_{H,V}$ and $\nabla_{H,V}$ have the property that they vanish on groups which do not have H as a section, and at H they take the value V.

Restriction and induction

Suppose $\mathcal{E} \subseteq \mathcal{D}$ are two section-closed sets of groups.
The restriction functor

$$\downarrow_{\mathcal{E}}^{\mathcal{D}} : \mathrm{Mack}_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D}) \to \mathrm{Mack}_R^{\mathcal{X},\mathcal{Y}}(\mathcal{E})$$

restricts the domain of a GDMF M from $\mathcal{D}$ to $\mathcal{E}$ giving a GDMF $M \downarrow_{\mathcal{E}}^{\mathcal{D}}$.
Restriction is exact.
It has a left adjoint $N \mapsto N \uparrow_{\mathcal{E}}^{\mathcal{D}} = \mu_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D}) \otimes_{\mu_R^{\mathcal{X},\mathcal{Y}}(\mathcal{E})} N$ called induction and a right adjoint
$N \mapsto M \Uparrow_{\mathcal{E}}^{\mathcal{D}} = \mathrm{Hom}_{\mu_R^{\mathcal{X},\mathcal{Y}}(\mathcal{E})}(\mu_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D}), N)$ called coinduction.
Question: are these the same?

Restriction is cutting by an idempotent

Suppose $\mathcal{E} \subseteq \mathcal{D}$ are two section-closed sets of groups. Suppose $\mathcal{E}$ is finite.
Consider the identity element

$$e_{\mathcal{E}} = \sum_{G \in \mathcal{E}} {}_G G_G \in \mu_R^{\mathcal{X},\mathcal{Y}}(\mathcal{E})$$

so that $\mu_R^{\mathcal{X},\mathcal{Y}}(\mathcal{E}) = e_{\mathcal{E}} \mu_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D}) e_{\mathcal{E}}$. Regarding M as a $\mu_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D})$-module its restriction is

$$M \downarrow_{\mathcal{E}}^{\mathcal{D}} = e_{\mathcal{E}} M.$$

This is a functor whose general properties were described by Green.

Restriction and induction of simples and projectives

Proposition

Let $\mathcal{E} \subseteq \mathcal{D}$ be sets of finite groups closed under taking sections and suppose that R is a field or a discrete valuation ring. Then

$$S^{\mathcal{D}}_{H,V} \downarrow_{\mathcal{E}} = \begin{cases} S^{\mathcal{E}}_{H,V} & \text{if } H \in \mathcal{E} \\ 0 & \text{if } H \notin \mathcal{E} \end{cases}$$

If H is a group in $\mathcal{E}$ then $P^{\mathcal{D}}_{H,V} \downarrow^{\mathcal{D}}_{\mathcal{E}} = P^{\mathcal{E}}_{H,V}$ and $P^{\mathcal{E}}_{H,V} \uparrow^{\mathcal{D}}_{\mathcal{E}} = P^{\mathcal{D}}_{H,V}$.

Definition of relative projectivity

Suppose $\mathcal{E} \subseteq \mathcal{D}$ are section closed sets of groups. A GDMF M defined on $\mathcal{D}$ is $\mathcal{E}$-projective if and only if the canonical counit homomorphism $M \downarrow_{\mathcal{E}}^{\mathcal{D}} \uparrow_{\mathcal{E}}^{\mathcal{D}} \rightarrow M$ is a split epimorphism. This condition is equivalent to a number of other standard conditions.

Indecomposability under induction and restriction

Proposition

Let $\mathcal{E} \subseteq \mathcal{D}$ be sets of finite groups closed under taking sections. Suppose M is a globally defined Mackey functor defined on $\mathcal{E}$ and L is a globally defined Mackey functor defined on $\mathcal{D}$ which is $\mathcal{E}$-projective. Then

1. $M \uparrow_{\mathcal{E}}^{\mathcal{D}} \downarrow_{\mathcal{E}}^{\mathcal{D}} \cong M$,
2. *if M is indecomposable then* $M \uparrow_{\mathcal{E}}^{\mathcal{D}}$ *is indecomposable,*
3. $L \cong L \downarrow_{\mathcal{E}}^{\mathcal{D}} \uparrow_{\mathcal{E}}^{\mathcal{D}}$,
4. *if L is indecomposable then* $L \downarrow_{\mathcal{E}}^{\mathcal{D}}$ *is indecomposable.*

The vertex of a GDMF

Theorem

Let M be a globally defined Mackey functor. There is a unique minimal set of groups $\mathcal{E}$, closed under taking sections, relative to which M is projective. Furthermore $M \cong M \downarrow_{\mathcal{E}}^{\mathcal{D}} \uparrow_{\mathcal{E}}^{\mathcal{D}}$, and $M \downarrow_{\mathcal{E}}^{\mathcal{D}}$ is (up to isomorphism) the only Mackey functor N defined on $\mathcal{E}$ with the property that $M \cong N \uparrow_{\mathcal{E}}^{\mathcal{D}}$.

Note that there is no condition on R or requirement that M be indecomposable. We call the minimal set $\mathcal{E}$ the vertex of M.

The vertices of some GDMFs

Theorem

Let H be a group and V a simple $R\,Out(H)$-module. The following all have vertex the set of sections of H.

- *The representable functor $Hom_{\mathcal{C}_R^{\mathcal{X},\mathcal{Y}}(\mathcal{D})}(H, \quad)$,*
- *the indecomposable projective functor $P_{H,V}$ (assuming R is a field or a discrete valuation ring),*
- *the standard functor $\Delta_{H,V}$.*

The End

Slides for this talk will be available at:

`htt://www.math.umn.edu/~webb`