Weight Theory for Modular Representations of Finite Groups

Peter Webb, Minnesota, September 8, 2005

Outline:

1. Character tables
2. Blocks of defect zero
3. Alperin's weight conjecture
4. The poset of p-subgroups
5. The orbit category of p-subgroups

1. Character tables

A *representation* of a group G is a group homomorphism $G \to GL(V)$ where V is a vector space over some field k (or more generally a module for some commutative ring). It is *simple*, or *irreducible*, if there are no proper subspaces invariant under the action of G.

If $k = \mathbb{C}$ we define the *character* of G to be the function $\chi_V : G \to \mathbb{C}$ which is $\chi_V(g) = \mathrm{tr}(g)$. It is constant on conjugacy classes. The *character table* is the matrix with entries $\chi_V(g)$ where V ranges over the simple representations over $\mathbb{C}$, g ranges over representatives of the conjugacy classes.

The table is square. The columns are orthogonal. The rows are orthogonal with respect to a particular inner product.

The degree of a character χ_V is $\chi_V(1) = \dim V$. The degrees of simple characters all divide $|G|$, the sum of their squares equals $|G|$.

Examples: J_1.

The character table of S_3, of order $6 = 2 \cdot 3$:

()	(12)	(123)
1	1	1
1	−1	1
2	0	−1

The character table of S_4, of order $24 = 8 \cdot 3$:

()	(12)	(12)(34)	(1234)	(123)
1	1	1	1	1
1	−1	1	−1	1
2	0	2	0	−1
3	−1	−1	1	0
3	1	−1	−1	0

The character table of $GL(3,2)$, of order $168 = 8 \cdot 3 \cdot 7$:

1	2	4	3	7A	7B
1	1	1	1	1	1
3	-1	1	0	α	$\overline{\alpha}$
3	-1	1	0	$\overline{\alpha}$	α
6	2	0	0	-1	-1
7	-1	-1	1	0	0
8	0	0	-1	1	1

Here $\alpha = \hat{\eta} + \hat{\eta}^2 + \hat{\eta}^4$ where $\hat{\eta} = e^{2\pi i/7}$.

2. Blocks of defect zero

Fix a group G, a prime p. A *p-block of defect zero* is a simple character χ for which $|G|_p \mid \chi(1)$.

THEOREM. *Let χ be a p-block of defect zero and $g \in G$ an element of order divisible by p (a p-singular element). Then $\chi(g) = 0$.*

Examples: J_1, S_3. Almost all the zeros in the character table of J_1 are accounted for by the fact that they occur in blocks of defect zero.

THEOREM. *Under reduction modulo p, p-blocks of defect zero correspond to simple projective kG-modules where k is an algebraically closed field of characteristic p, and also to matrix ring summands of kG.*

The blocks of defect zero exactly correspond to the largest semisimple ring direct summand of kG.

Not all groups have p-blocks of defect zero; for example p-groups of order > 1 and S_4 at $p = 2$ do not. If $p \not| \, |G|$ then every simple character is a p-block of defect zero.

3. Alperin's weight conjecture

Let G be a finite group and k an algebraically closed field of characteristic p. Alperin defines a weight to be a pair (H, V) where $H \leq G$ is a p-subgroup and V is a p-block of defect zero of $N_G(H)/H$. We take H up to conjugacy and V up to isomorphism.

CONJECTURE (J. Alperin). *The number of weights of G equals the number of conjugacy classes of elements of G of order prime to p.*

Elements of order prime to p are called p-regular.

THEOREM (Brauer). *The number of simple kG-modules equals the number of p-regular elements.*

Thus weights conjecturally biject with simple kG-modules. However, we expect there not to be a canonical bijection except in special cases.

Examples:

	subgroup H	$N_G(H)/H$	# blocks of def. 0
S_3 :	1	S_3	1
	C_2	1	$+1$
			$= 2$ weights
S_4 :	1	S_4	0
	V	S_3	$+1$
	D_8	1	$+1$
			$= 2$ weights
$GL(3,2)$:	1	$GL(3,2)$	1
	V_1	S_3	$+1$
	V_2	S_3	$+1$
	D_8	1	$+1$
			$= 4$ weights

Significance of the conjecture and local theory

Why is this important? The conjecture connects two sets which otherwise are seemingly unrelated. The simple modules are enumerated by Brauer's theorem and this is easy. The blocks of defect zero are notoriously difficult to pin down in generality. There is a theorem of Robinson which gives an expression for their number, but it is not explicit and does not yield a proof of the conjecture. The fact that the conjecture appears to be true suggests the possibility of some structure which has not so far been observed or exploited, and whatever such structure is it is probably interesting. The conjecture is part of a philosophy in which properties of the representations of the whole group are determined from local information, that is, conjugation information from subgroups which have the form $N_G(H)$ where H is a p-group. This philosophy underlies Alperin's fusion theorem, the amalgam method which is part of the current proof of the classification of finite simple groups, much work in modular representation theory (including Broué's conjecture) and the study of p-local finite groups.

The conjecture is known to be true for

1. p-solvable groups (Okuyama)
2. Groups of Lie type in defining characteristic.
3. Symmetric groups
4. Many sporadic simple groups

There is a program due to Dade which purports to reduce a stronger version of the conjecture to almost simple groups (a central extension of a group of automorphisms of a simple group). However the argument which makes this reduction has not appeared. Also, the checking to verify the stronger conjecture seems impossible in any reasonable amount of time.

There have been many reformulations of the conjecture which have come about in attempts to prove it. One of them, due to Knörr and Robinson reformulates the conjecture as identities arising from a simplicial complex homotopy equivalent to the p-subgroups complex.

4. The poset of p-subgroups

THEOREM (Knörr-Robinson). *Assume the validity of Alperin's weight conjecture. Let Δ be the order complex of the poset of p-subgroups of G (including the empty simplex in degree -1). Then the number of blocks of defect zero is*

$$-\sum_{\sigma \in \Delta/G} (-1)^{\dim \sigma} \# \text{ conjugacy classes of } G_\sigma.$$

It also equals

$$-\sum_{\sigma \in \Delta/G} (-1)^{\dim \sigma} \# \text{ } p\text{-regular conjugacy classes of } G_\sigma.$$

Conversely the validity of these statements for all G implies Alperin's conjecture.

Example: $GL(3,2)$ at $p = 2$:

$$1 = 6 - 5 - 5 + 5$$

5. The orbit category of p-subgroups

Let $\mathcal{O}$ be the category whose objects are the G-sets G/H where $H \leq G$ is a p-subgroup and the morphisms $G/H \to G/K$ are the G-equivariant maps. All morphisms are epimorphisms and $\mathrm{Aut}(G/H) \cong N_G(H)/H$.

A representation of $\mathcal{O}$ over a ring R is a functor $F : \mathcal{O} \to R$-mod. For example group cohomology $F(G/K) = H^*(K, V)$ using corestriction and conjugation maps provides a representation of the orbit category. More generally, any Mackey functor gives such an example. Contravariant functors on $\mathcal{O}$ (representations of the opposite category) are called Bredon coefficient systems and were introduced by Bredon in the 1960s in defining Bredon cohomology.

Fei Xu is developing certain aspects of the representation theory of such categories in abstract, producing a theory analogous to that of group representations.

The simple representations of $\mathcal{O}$ are in bijection with pairs $(G/H, V)$ where V is a simple representation of $\mathrm{Aut}(G/H) \cong N_G(H)/H$. There is a preorder on the simples specified by $(G/H, V) \leq (G/K, W) \Leftrightarrow |H| \leq |K|$.

THEOREM (Webb). *Representations of $\mathcal{O}$ over any field k are standardly stratified with respect to the preorder.*

Standard stratification was introduced by Cline-Parshall-Scott and also by Dlab in the mid 1990s. It means that there are certain 'standard' representations $\Delta_{H,V}$ which play the role of Verma modules, and proper costandard modules $\bar{\nabla}_{H,V}$ which play the role of their duals so that projective representations have a standard filtration, injectives have a proper costandard filtration with certain properties. In this theory the representations which have both a standard and a proper costandard filtration are called *tilting modules.*

THEOREM (Webb). *Consider representations of $\mathcal{O}$ over a field k of characteristic p. Then $\Delta_{H,V}$ is a tilting module if and only if $H = 1$. $\bar{\nabla}_{H,V}$ is injective if and only if (H, V) is a weight.*

The idea here is that the framework of stratification will provide the extra structure which lies behind Alperin's conjecture. We look for a generalization of the usual Lie theory, in which the role of the Cartan subalgebra is played by $\bigoplus k[N_G(H)/H]$ where H ranges over p-subgroups of G, and Alperin's weights are the simple representations of the largest semisimple ring direct summand of this algebra. Mackey functors also provide possibilities for such a theory.