Constructing Resolutions for p-groups

Peter Webb

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email: webb@math.umn.edu
http://www.math.umn.edu/∼webb

The text of this talk can be obtained from the above site.

Outline of talk

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1. Introduction

I will show how to construct a minimal resolution of k over kG, wneh G is a p-group and k is a field of characteristic p, without finding preimages of elements, inverting matrices or taking intersections of subspaces. The only operations used are the basic ones of matrix addition and multiplication.

Some ways to compute the (co)homology of a finite p-group:

Method	Advantages/Disadvantages
LHS spectral sequence	It's hard to program.
Compute a resolution by computing projective covers and kernels using matrices	It's easy to program. We get a minimal resolution. It becomes slow.
Compute a resolution from a presentation of kG using Groebner basis methods	Harder to program. We get a minimal resolution. It becomes slow.
Bar resolution, construct BG	These are easy to program but too big.
Wall's resolution	

This list of methods is by no means complete. We mention also the method used by Rusin and special techniques for particular groups.

2. Constructing a non-minimal resolution in characteristic 2

Let G be a 2-group, H a subgroup of index 2 in G,
k a field of characteristic 2.
Suppose we are given a resolution

$$k \leftarrow \overbrace{Q_0 \leftarrow Q_1 \leftarrow \cdots}^{\mathcal{Q}}$$

of k over kH.
We will construct a resolution

$$k \leftarrow \overbrace{P_0 \leftarrow P_1 \leftarrow \cdots}^{\mathcal{P}}$$

over kG.

At the moment we don't know what $\mathcal{P}$ is, but let us pretend that we do. Let $g \in G - H$ and consider the extension

$$0 \longrightarrow k \xrightarrow{1-g} k \uparrow_H^G \longrightarrow k \longrightarrow 0$$

associated to H. We may lift maps from $\mathcal{P}$ as follows

$$\begin{array}{ccccccccc}
k & \longleftarrow & P_0 & \xleftarrow{e_1} & P_1 & \xleftarrow{e_2} & P_2 & \xleftarrow{e_3} & \\
{\scriptstyle 1-g}\downarrow & & {\scriptstyle u_0}\downarrow & & {\scriptstyle u_1}\downarrow & & {\scriptstyle u_2}\downarrow & & \\
k \uparrow_H^G & \longleftarrow & Q_0 \uparrow_H^G & \xleftarrow{d_1} & Q_1 \uparrow_H^G & \xleftarrow{d_2} & Q_2 \uparrow_H^G & \xleftarrow{d_3} & \\
\downarrow & & & & & & & & \\
k & & & & & & & &
\end{array}$$

to get a map of complexes $u : \mathcal{P} \to \mathcal{Q} \uparrow_H^G$.
We form the mapping cone of u.
This is a complex $C(u)$:

$$Q_0 \uparrow_H^G \xleftarrow{\begin{pmatrix} -d_1 & u_0 \end{pmatrix}} Q_1 \uparrow_H^G \oplus P_0 \xleftarrow{\begin{pmatrix} -d_2 & u_1 \\ 0 & e_1 \end{pmatrix}} Q_2 \uparrow_H^G \oplus P_1 \xleftarrow{\begin{pmatrix} -d_3 & u_2 \\ 0 & e_2 \end{pmatrix}}$$

PROPOSITION/EXERCISE. *$C(u)$ is a projective resolution of k.*

Proof. The modules here are projective. By looking at the long exact sequence in homology associated to the exact sequence

$$0 \to \mathcal{Q} \uparrow_H^G \to C(u) \to \mathcal{P}[1] \to 0$$

which ends

$$\cdots \to 0 \to 0 \to k \to k \uparrow_H^G \to H_0(C(u)) \to 0$$

we see that $C(u)$ is a projective resolution of k. □

At this point, one idea is that we may now *define $\mathcal{P}$ to be $C(u)$.*
$\mathcal{P}$ appears as the top and middle of a diagram as follows.

$$\begin{array}{ccccc}
Q_0 \uparrow_H^G & \overset{\left(\begin{smallmatrix} -d_1 & u_0 \end{smallmatrix}\right)}{\longleftarrow} & Q_1 \uparrow_H^G \oplus P_0 & \overset{\left(\begin{smallmatrix} -d_2 & u_1 \\ 0 & e_1 \end{smallmatrix}\right)}{\longleftarrow} & Q_2 \uparrow_H^G \oplus P_1 \\
\| & & \| & & \| \\
P_0 & \overset{e_1}{\longleftarrow} & P_1 & \overset{e_2}{\longleftarrow} & P_2 \\
{\scriptstyle u_0}\downarrow & & {\scriptstyle u_1}\downarrow & & {\scriptstyle u_2}\downarrow \\
Q_0 \uparrow_H^G & \overset{d_1}{\longleftarrow} & Q_1 \uparrow_H^G & \overset{d_2}{\longleftarrow} & Q_2 \uparrow_H^G
\end{array}$$

We may take $P_0 = Q_0 \uparrow_H^G = kG$, $u_0 = 1 - g$, and no matter what the e_i are, the first map of $C(u)$ is $(-d_1, u_0)$. Thus there certainly is a resolution of k starting this way, and we might as well define $e_1 = (-d_1, u_0)$. Having made this definition, the second map in $C(u)$ is defined as shown, no matter what $e_2, e_3, \ldots$ are, so we might as well define e_2 to be this map. Continuing this argument shows that a resolution with the property $\mathcal{P} = C(u)$ exists, and it may be computed in an inductive fashion.
We see in this resolution that

$$P_n = Q_n \uparrow_H^G \oplus Q_{n-1} \uparrow_H^G \oplus \cdots \oplus Q_0 \uparrow_H^G .$$

We also see that information about the resolution up to stage n is entirely encoded in the single matrix

$$e_n = \begin{pmatrix} -d_n & u_{n-1} \\ 0 & e_{n-1} \end{pmatrix},$$

since this includes the e_i with $i \le n$ as submatrices, and also the u_i with $i \le n-1$ are stored as the right-hand parts of the rows.
This resolution has some quite interesting properties, but for the moment we consider only algorithmic questions.

We have produced a resolution over kG from the following information

1. a resolution over kH,
2. the lifts u_n.

In particular, we never compute kernels. The point not so far explained is how we compute the u_n. One approach is to solve a system of linear equations. We will present a different approach.

Partial algorithm to compute a resolution as a mapping cone

1. We suppose we have a free resolution $\mathcal{Q}$ of k over kH with maps given as matrices with entries in kH.

2. We form the induced resolution $\mathcal{Q} \uparrow_H^G$. This has terms which are free kG-modules of the same rank as in $\mathcal{Q}$, and the matrices giving the maps are the same as for $\mathcal{Q}$.

3. Start things off by taking $u_0 = 1 - g$ where g lies in $G - H$.

4. At stage n, when u_{n-1} has been computed, form the matrix

$$e_n = \begin{pmatrix} -d_n & u_{n-1} \\ 0 & e_{n-1} \end{pmatrix}.$$

5. Compute the matrix which represents u_n.

6. ... repeat.

After some examples we will show how the lifts u_n may be computed without solving equations.

3. Examples

Example 1. $G = C_4 = \langle g \rangle$ the cyclic group of order 4, $H = \langle g^2 \rangle$.
We start with a resolution $\mathcal{Q}$ of k over kH:

$$(0 \leftarrow k \leftarrow) \quad Q_0 \xleftarrow{1+g^2} Q_1 \xleftarrow{1+g^2} Q_2 \xleftarrow{1+g^2} \leftarrow \cdots$$

where each Q_i is a copy of kH. Inducing this to G we obtain

$$(0 \leftarrow k \uparrow_H^G \leftarrow) \quad Q_0 \uparrow_H^G \xleftarrow{1+g^2} Q_1 \uparrow_H^G \xleftarrow{1+g^2} Q_2 \uparrow_H^G \xleftarrow{1+g^2} \leftarrow \cdots$$

The labels stay the same on induction. The construction of the resolution $\mathcal{P}$ over kG is now

$$\begin{array}{ccccc}
P_0 & \xleftarrow{\begin{pmatrix} 1+g^2 & 1+g \end{pmatrix}} & P_1 & \xleftarrow{\begin{pmatrix} 1+g^2 & 1+g & 1 \\ 0 & 1+g^2 & 1+g \end{pmatrix}} & P_2 \\
{\scriptstyle 1+g}\downarrow & & {\scriptstyle \begin{pmatrix} 1+g & 1 \end{pmatrix}}\downarrow & & {\scriptstyle \begin{pmatrix} 1+g & 0 & 0 \end{pmatrix}}\downarrow \\
Q_0 \uparrow_H^G & \xleftarrow{1+g^2} & Q_1 \uparrow_H^G & \xleftarrow{1+g^2} & Q_2 \uparrow_H^G
\end{array}$$

where $P_n = Q_n \uparrow_H^G \oplus \cdots \oplus Q_0 \uparrow_H^G \cong kG^{n+1}$.

The general form of the differential e_n is

$$e_n = \begin{pmatrix}
\ddots & & & & & & \vdots \\
& 1+g^2 & 1+g & 0 & 0 & 0 & 0 \\
& 0 & 1+g^2 & 1+g & 1 & 0 & 0 \\
& 0 & 0 & 1+g^2 & 1+g & 0 & 0 \\
& 0 & 0 & 0 & 1+g^2 & 1+g & 1 \\
\cdots & 0 & 0 & 0 & 0 & 1+g^2 & 1+g
\end{pmatrix}.$$

Reduced modulo the radical of kG this is

$$\begin{pmatrix} \ddots & & & & & & \vdots \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 1 \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The ranks of the successive differentials after reduction modulo the radical of kG are $0, 1, 1, 2, 2, 3, 3, \ldots$. Since these have domains of dimensions $1, 2, 3, 4, 5, \ldots$ we obtain that $\dim H^i(C_4, k) = 1$ for all i. By examining these matrices we also see that a minimal resolution may be obtained by removing $Q_0 \uparrow_H^G$ from P_i for all $i \geq 2$, together with the image of $Q_0 \uparrow_H^G$ under e_i, removing $Q_2 \uparrow_H^G$ from P_i for all $i \geq 4$, together with the image of $Q_2 \uparrow_H^G$ under e_i, and so on.

Example 2. The quaternion group of order 8.
$G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, $H = \langle i \rangle$.

We start with a resolution $\mathcal{Q}$ of k over kH:

$$(0 \leftarrow k \leftarrow) \quad Q_0 \xleftarrow{1+i} Q_1 \xleftarrow{(1+i)^3} Q_2 \xleftarrow{1+i} \leftarrow \cdots$$

where each Q_i is a copy of kH. Inducing this to G we obtain

$$(0 \leftarrow k \uparrow_H^G \leftarrow) \quad Q_0 \uparrow_H^G \xleftarrow{1+i} Q_1 \uparrow_H^G \xleftarrow{(1+i)^3} Q_2 \uparrow_H^G \xleftarrow{1+i} \leftarrow \cdots$$

The construction of the resolution $\mathcal{P}$ over kG is now

$$\begin{array}{ccccc} P_0 & \xleftarrow{\begin{pmatrix} 1+i & 1+j \end{pmatrix}} & P_1 & \xleftarrow{\begin{pmatrix} (1+i)^3 & 1+k & 1+i \\ 0 & 1+i & 1+j \end{pmatrix}} & P_2 \\ {\scriptstyle 1+j}\big\downarrow & & {\scriptstyle \begin{pmatrix} 1+k & 1+i \end{pmatrix}}\big\downarrow & & {\scriptstyle \begin{pmatrix} 1+j & 0 & j \end{pmatrix}}\big\downarrow \\ Q_0 \uparrow_H^G & \xleftarrow{1+i} & Q_1 \uparrow_H^G & \xleftarrow{(1+i)^3} & Q_2 \uparrow_H^G \end{array}$$

where $P_n = Q_n \uparrow_H^G \oplus \cdots \oplus Q_0 \uparrow_H^G \cong kG^{n+1}$. In order to do these computations we have to solve systems equations in the group ring, such as

$$(1+i)(1+j) = x(1+i)$$
$$(1+j)(1+j) = y(1+i)$$

with solution

$$(x, y) = (1+k, 1+i)$$

which gives u_1 and

$$\begin{aligned}(1+i)^3(1+k) &= r(1+i)^3\\ (1+k)^2+(1+i)^2 &= s(1+i)^3\\ (1+i)(1+k)+(1+j)(1+i) &= t(1+i)^3\end{aligned}$$

with solution

$$(r,s,t) = (1+j,0,j)$$

which gives u_2; and so on.

The general form of the differential e_n is a matrix with n rows

$$e_n = \begin{pmatrix} \ddots & & & & & & & & & \vdots \\ & (1+i)^3 & 1+k & 1+i & k & 0 & k & 0 & k & 0 \\ & 0 & 1+i & 1+j & 0 & j & 0 & j & 0 & j \\ & 0 & 0 & (1+i)^3 & 1+k & 1+i & 0 & 0 & k & 0 \\ & 0 & 0 & 0 & 1+i & 1+j & 0 & 0 & 0 & j \\ & 0 & 0 & 0 & 0 & (1+i)^3 & 1+k & 1+i & k & 0 \\ & 0 & 0 & 0 & 0 & 0 & 1+i & 1+j & 0 & j \\ & 0 & 0 & 0 & 0 & 0 & 0 & (1+i)^3 & 1+k & 1+i \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+i & 1+j \end{pmatrix}$$

which reduced modulo the radical of kG is

$$\begin{pmatrix} \ddots & & & & & & & & & \vdots \\ & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From this we obtain that the dimensions of $H^i(Q_8,k)$ are $1,2,2,1,1,2,2,\ldots$

4. Splitting the resolution

Definition. A splitting of a complex $\cdots \leftarrow C_{i-1} \overset{d_i}{\leftarrow} C_i \overset{d_{i+1}}{\leftarrow} C_{i+1} \leftarrow \cdots$ is a family of maps $s_i : C_i \to C_{i+1}$ of degree $+1$ so that $dsd = d$ always holds.

PROPOSITION/EXERCISE ('Subgroup Complexes', Arcata Proceedings p.363). *The following are equivalent:*
(1) $C.$ has a splitting,
(2) $C_i \cong d(C_i) \oplus H_i(C.) \oplus d(C_{i+1})$ for all i in such a way that d is an isomorphism on the summands $d(C_i)$ and is zero on the other summands.

Given a projective kG-resolution of k, it has no G-equivariant splitting, since this would force k to be projective and $G = 1$. It does, however, have non-equivariant splittings.
A non-equivariant splitting of $\mathcal{Q} \uparrow_H^G$ allows us to compute the lifts u_n:

$$\begin{array}{ccccccc} \overset{e_{n-1}}{\longleftarrow} & P_{n-1} & \overset{e_n}{\longleftarrow} & P_n & \overset{e_{n+1}}{\longleftarrow} \\ & {\scriptstyle u_{n-1}}\big\downarrow & & {\scriptstyle u_n}\big\downarrow & \\ \overset{s_{n-1}}{\longrightarrow} & Q_{n-1} \uparrow_H^G & \overset{s_n}{\longrightarrow} & Q_n \uparrow_H^G & \overset{s_{n+1}}{\longrightarrow} \end{array}$$

For each free generator x_j of P_n, the assignment $x \mapsto s_n u_{n-1} e_n(x_j)$ extends uniquely to a kG-module map $u_n : P_n \to Q_n$, and now $u_{n-1} d_n = d_n u_n$.

PROPOSITION. *Suppose that $(\mathcal{Q} \uparrow_H^G, d, s)$ is a kG-projective resolution of $k \uparrow_H^G$ with differential d and non-equivariant splitting s. Suppose that $u_n : P_n \to Q_n \uparrow_H^G$ is defined as above, using the splitting s_n. Then*

$$\sigma_n = \begin{pmatrix} -s_n & s_n u_n \sigma_{n-1} \\ 0 & \sigma_{n-1} \end{pmatrix} : P_{n-1} \to P_n$$

when $n \geq 2$, and with σ_1 defined suitably (below), is a splitting of $\mathcal{P} = C.(u)$.

Note that $P_{n-1} \to P_n$ may be written as $Q_{n-1} \uparrow \oplus P_{n-2} \to Q_n \uparrow \oplus P_{n-1}$ and the matrix refers to this decomposition.
Proof. The plan is to define σ_1 directly so that $e_1 \sigma_1 e_1 = e_1$. We then verify that $e_n \sigma_n e_n = e_n$ by multiplying matrices and using induction on n.
The definition of $\sigma_1 : kG \to Q_0 \uparrow_H^G \oplus kG$ is as follows. Here $g \in G - H$ and we let x denote an arbitrary element of the augmentation ideal of kH.

$$\sigma_1 : \begin{cases} 1 \mapsto (0,0)^T \\ g \to (0,-1)^T \\ x \mapsto (s_1(x), 0)^T \\ gx \mapsto (gs_1(x), 0)^T. \end{cases}$$

□

Algorithm to compute a polynomial growth resolution using only matrix multiplications

We suppose $H \leq G$ with $|G : H| = 2$, and let $g \in G - H$. Inductively we suppose we are given a resolution $\mathcal{Q}$ of k by free kH-modules, together with a non-equivariant splitting of $\mathcal{Q}$. We produce a resolution $\mathcal{P}$ of k by free kG-modules, together with a non-equivariant splitting of $\mathcal{P}$.

1. Construct

$$\mathcal{Q}\uparrow_H^G = \quad Q_0\uparrow_H^G \underset{s_1}{\overset{d_1}{\leftrightarrows}} Q_1\uparrow_H^G \underset{s_2}{\overset{d_2}{\leftrightarrows}} \cdots$$

by an induction construction. In terms of matrices this means duplicating the differential and splitting.

2. Given e_{n-1} and u_{n-1} construct $e_n = \begin{pmatrix} -d_n & u_{n-1} \\ 0 & e_{n-1} \end{pmatrix}$, and also u_n extending $s_n u_{n-1} d_n$ on the free generators.

3. Construct splittings $\sigma_n : P_{n-1} \to P_n$ as $\sigma_n = \begin{pmatrix} -s_n & s_n u_n \sigma_{n-1} \\ 0 & \sigma_{n-1} \end{pmatrix} : P_{n-1} \to P_n$.

Disadvantage of this algorithm: the ranks of the P_n become large, so that ordinary matrix multiplication becomes slow.

Advantage of this algorithm: the matrices e_n, and especially the σ_n, are sparse. Storing these as sparse matrices and using sparse multiplication may give an efficient way to compute a resolution.

5. Examples

$C_2 \times C_2$: the degree 4 boundary map and splitting:

```
gap> MatPrint(c2xc2info.res[4]);
[
[ [ 1, 0, 1, 0 ], [ 0, 0, 0, 0 ], [ 0, 0, 0, 0 ], [ 0, 0, 0, 0 ] ],
[ [ 1, 1, 0, 0 ], [ 1, 0, 1, 0 ], [ 0, 0, 0, 0 ], [ 0, 0, 0, 0 ] ],
[ [ 0, 0, 0, 0 ], [ 1, 1, 0, 0 ], [ 1, 0, 1, 0 ], [ 0, 0, 0, 0 ] ],
[ [ 0, 0, 0, 0 ], [ 0, 0, 0, 0 ], [ 1, 1, 0, 0 ], [ 1, 0, 1, 0 ] ],
[ [ 0, 0, 0, 0 ], [ 0, 0, 0, 0 ], [ 0, 0, 0, 0 ], [ 1, 1, 0, 0 ] ] ]
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[
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[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
[ 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
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[ [ 16, 20 ], [ [ ], [ ], [ 1 ], [ 2 ], [ ], [ ], [ 5 ], [ 6 ], [ ], [ ], [ 9 ], [ 10 ],
[ ], [ 17 ], [ 13 ], [ 14, 17 ] ] ]
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D_8: the degree 3 boundary map and splitting:

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gap> MatPrint(d8info.res[3]);
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[ [ 24, 32 ], [ [ ], [ 17 ], [ 24 ], [ 19, 24 ], [ 17, 18 ], [ 17, 18, 21 ], [ 19, 20, 24 ],
[ ], [ ], [ ], [ 9 ], [ 2, 10, 19, 24, 27, 32 ], [ 3, 17, 18, 25, 26 ],
[ 8, 18, 21, 26, 29 ], [ 1, 9, 19, 20, 27, 28 ], [ 10 ], [ ], [ 24 ], [ 17, 25 ],
[ 17, 25 ], [ 24 ], [ 2, 17, 19, 24 ], [ 3, 17, 18, 24, 25 ], [ 25 ] ] ]
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6. Computing a minimal resolution.

This could be done by taking some other resolution, for example the mapping cone resolution just described, or the bar resolution, and stripping off free summands to make it minimal. In this section we explain how to do this without letting the resolution get big in the first place.

We will compute a minimal resolution $\mathcal{P} = \quad k \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$ which is part of a diagram

$$\begin{array}{ccccccc}
C_0 & \stackrel{e_1}{\longleftarrow} & C_1 & \stackrel{e_2}{\longleftarrow} & C_2 & \stackrel{e_3}{\longleftarrow} \\
\| & & {\scriptstyle w_1}\downarrow & & {\scriptstyle w_2}\downarrow & \\
P_0 & \stackrel{\delta_1}{\longleftarrow} & P_1 & \stackrel{\delta_2}{\longleftarrow} & P_2 & \stackrel{\delta_3}{\longleftarrow} \\
{\scriptstyle u_0}\downarrow & & {\scriptstyle u_1}\downarrow & & {\scriptstyle u_2}\downarrow & \\
Q_0 \uparrow_H^G & \stackrel{d_1}{\longleftarrow} & Q_1 \uparrow_H^G & \stackrel{d_2}{\longleftarrow} & Q_2 \uparrow_H^G & \stackrel{d_3}{\longleftarrow}
\end{array}$$

Here $C. = C.(u)$ is the mapping cone of u, and it is also a projective resolution of k.

We start with $P_0 = C_0 = kG$. Inductively, suppose that $P_0 \leftarrow \cdots \leftarrow P_{n-1}$ has been constructed, minimal up to P_{n-2}, not necessarily minimal at P_{n-1}. Suppose that maps $w_1, \ldots, w_{n-1}$ have been constructed, and they are all surjective. Then $w_{n-1}e_n : C_n \to P_{n-1}$ is surjective and

PROPOSITION/EXERCISE. *P_{n-1} is minimal $\Leftrightarrow$ $w_{n-1}e_n$ is represented by a matrix of elements of kG with no unit entries.*

In case there are unit entries, they may easily be used to factor kG summands from P_{n-1}, making it smaller, in turn making C_n smaller. A new map w_{n-1} is computed by composition with the factor map, and it is surjective.

P_{n-1} is now minimal. The corresponding kG-summands of C_n may be factored from C_n to give a surjection $w_n : C_n \to P_n$, which we use for the next iteration of the algorithm.

It remains to compute splittings of $\mathcal{P}$. We have splittings σ_n of $C.$, and we may use them to compute a chain map $v : \mathcal{P} \to C.$ which is a chain homotopy inverse to w. We now use v and w to transfer the splittings of $C.$ to give splittings of P. This idea appears in 'Subgroup complexes', Arcata Proceedings, page 364.

7. General construction of the resolution

This works whenever we have a subgroup $H \triangleleft G$ with $G/H = \langle gH \rangle$ cyclic, over any ground ring R. It is convenient but not necessary to assume $G = H \rtimes \langle g \rangle$ for some $g \in G$.

Suppose we have a resolution

$$R \leftarrow \overbrace{Q_0 \leftarrow Q_1 \leftarrow \cdots}^{\mathcal{Q}}$$

over RH, $Q_0 = RH$. We construct a resolution

$$R \leftarrow \overbrace{P_0 \leftarrow P_1 \leftarrow \cdots}^{\mathcal{P}}$$

over RG.
Consider

$$0 \to R \xrightarrow{\sum g^i} R \uparrow_H^G \xrightarrow{g-1} R \uparrow_H^G \xrightarrow{1} R \to 0$$

and lift $g-1$ as follows:

$$\begin{array}{ccccccc} R\uparrow_H^G & \longleftarrow & Q_0\uparrow_H^G & \xleftarrow{d_1} & Q_1\uparrow_H^G & \xleftarrow{d_2} & \cdots \\ {\scriptstyle g-1}\downarrow & & {\scriptstyle u_0}\downarrow & & {\scriptstyle u_1}\downarrow & & \\ R\uparrow_H^G & \longleftarrow & Q_0\uparrow_H^G & \xleftarrow{d_1} & Q_1\uparrow_H^G & \xleftarrow{d_2} & \cdots \end{array}$$

Form the mapping cone $C(u)$:

$$Q_0\uparrow_H^G \xleftarrow{\begin{pmatrix} -d_1 & u_0 \end{pmatrix}} Q_1\uparrow_H^G \oplus Q_0\uparrow_H^G \xleftarrow{\begin{pmatrix} -d_2 & u_1 \\ 0 & d_1 \end{pmatrix}} Q_2\uparrow_H^G \oplus Q_1\uparrow_H^G \xleftarrow{\begin{pmatrix} -d_3 & u_2 \\ 0 & d_2 \end{pmatrix}}$$

LEMMA.

$$H_i(C(u)) = \begin{cases} R & \text{if } i = 0 \text{ or } 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Use the long exact sequence in homology. □

Whenever $\mathcal{P}$ is a projective resolution of R we may use it to kill $H_1(C(u))$ as follows. We choose a map $v_1 : P_0 \to C_1(u)$ extending the map $P_0 \to H_1(C(u))$ and lift it to a map of complexes $v : \mathcal{P}[1] \to C(u)$. When $G = H \rtimes \langle g \rangle$ this map be represented as shown:

$$\begin{array}{ccccc} & & P_0 & \xleftarrow{e_1} & P_1 \\ & & {\scriptstyle \begin{pmatrix} 0 \\ \sum g^i \end{pmatrix}}\downarrow & & {\scriptstyle v_2}\downarrow \\ Q_0\uparrow_H^G & \xleftarrow{\begin{pmatrix} -d_1 & u_0 \end{pmatrix}} & Q_1\uparrow_H^G \oplus Q_0\uparrow_H^G & \xleftarrow{\begin{pmatrix} -d_2 & u_1 \\ 0 & d_1 \end{pmatrix}} & Q_2\uparrow_H^G \oplus Q_1\uparrow_H^G \end{array}$$

Form the mapping cone $C(v)$.

LEMMA. *$C(v)$ is a projective resolution of R over RG.*

Proof. Use the long exact sequence in homology. □

We now define $\mathcal{P}$ to be $C(v)$. Specifically

$$\begin{array}{ccccc}
P_0 & \xleftarrow{e_1} & P_1 & \xleftarrow{e_2} & \\
\| & & \| & & \\
Q_0\uparrow_H^G & \xleftarrow{\begin{pmatrix} d_1 & -u_0 \end{pmatrix}} & Q_1\uparrow_H^G \oplus Q_0\uparrow_H^G & \xleftarrow{\begin{pmatrix} d_2 & -u_1 & v_1^{1,0} \\ 0 & -d_1 & v_1^{0,0} \end{pmatrix}} &
\end{array}$$

$$\begin{array}{ccc}
P_2 & \xleftarrow{e_3} & \\
\| & & \\
Q_2\uparrow_H^G \oplus Q_1\uparrow_H^G \oplus P_0 & \xleftarrow{\begin{pmatrix} d_3 & -u_2 & v_2^{2,\cdot} \\ 0 & -d_2 & v_2^{1,\cdot} \\ 0 & 0 & e_1 \end{pmatrix}} &
\end{array}$$

From this we deduce

$$P_n = Q_n\uparrow_H^G \oplus Q_{n-1}\uparrow_H^G \oplus \cdots \oplus Q_0\uparrow_H^G$$

and

$$e_1 = \begin{pmatrix} d_1 & -u_0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} d_2 & -u_1 & v_1^{1,0} \\ 0 & -d_1 & v_1^{0,0} \end{pmatrix}.$$

The expression for e_3 has e_1 as an entry, but since e_1 is already determined we may substitute this to get

$$e_3 = \begin{pmatrix} d_3 & -u_2 & v_2^{2,1} & v_2^{2,0} \\ 0 & -d_2 & v_2^{1,1} & v_2^{1,0} \\ 0 & 0 & d_1 & -u_0 \end{pmatrix}.$$

Similarly

$$e_4 = \begin{pmatrix} d_4 & -u_3 & v_3^{3,2} & v_3^{3,1} & v_3^{3,0} \\ 0 & -d_3 & v_3^{2,2} & v_3^{2,1} & v_3^{2,0} \\ 0 & 0 & d_2 & -u_1 & v_1^{1,0} \\ 0 & 0 & 0 & -d_1 & v_1^{0,0} \end{pmatrix},$$

and so on.

8. Properties of the resolution

Property (i) The resolution replicates itself.
What we mean here is that there is a short exact sequence of complexes

$$0 \to \mathcal{Q}\uparrow_H^G \to \mathcal{P} \to \mathcal{P}[1] \to 0$$

so that the whole of the resolution can be seen as a quotient complex of itself starting in dimension 1. This may be pictured by drawing the resolution as an array, which in the case of the Q_8 example is

$$\begin{array}{cccccccc}
\vdots & & \vdots & & \vdots & & \vdots & \\
Q_0\uparrow_H^G & \xleftarrow{1+i} & Q_1\uparrow_H^G & \xleftarrow{(1+i)^3} & Q_2\uparrow_H^G & \xleftarrow{1+i} & Q_3\uparrow_H^G & \cdots \\
{\scriptstyle 1+j}\big\downarrow & & {\scriptstyle 1+k}\big\downarrow & & {\scriptstyle 1+j}\big\downarrow & & {\scriptstyle 1+k}\big\downarrow & \\
Q_0\uparrow_H^G & \xleftarrow{1+i} & Q_1\uparrow_H^G & \xleftarrow{(1+i)^3} & Q_2\uparrow_H^G & \xleftarrow{1+i} & Q_3\uparrow_H^G & \cdots \\
{\scriptstyle 1+j}\big\downarrow & & {\scriptstyle 1+k}\big\downarrow & & {\scriptstyle 1+j}\big\downarrow & & {\scriptstyle 1+k}\big\downarrow & \\
Q_0\uparrow_H^G & \xleftarrow{1+i} & Q_1\uparrow_H^G & \xleftarrow{(1+i)^3} & Q_2\uparrow_H^G & \xleftarrow{1+i} & Q_3\uparrow_H^G & \cdots \\
{\scriptstyle 1+j}\big\downarrow & & {\scriptstyle 1+k}\big\downarrow & & {\scriptstyle 1+j}\big\downarrow & & {\scriptstyle 1+k}\big\downarrow & \\
Q_0\uparrow_H^G & \xleftarrow{1+i} & Q_1\uparrow_H^G & \xleftarrow{(1+i)^3} & Q_2\uparrow_H^G & \xleftarrow{1+i} & Q_3\uparrow_H^G & \cdots
\end{array}$$

This diagram is not supposed to be a double complex!

Not all the component morphisms are shown!
There is a morphism $1+i$ from the Q_0 summand of P_2 to the Q_1 summand of P_1, and another such from the Q_0 summand of P_3 to the Q_1 summand of P_2, and so on with copies of the same map appearing in all vertically higher places. Also there are maps j from the Q_0 summand of P_3 to the Q_2 summand of P_2, k from the Q_1 summand of P_4 to the Q_3 summand of P_3, and $1+i$ from the Q_2 summand of P_4 to the Q_3 summand of P_3.

All of these maps replicate vertically.

Property (ii)
The powers of the extension determined by the subgroup H are easily identified.

Let $\alpha \in \mathrm{Ext}^1_{kG}(k,k)$ be the class of the extension

$$0 \to k \to k\uparrow_H^G \to k \to 0$$

determined by H.

PROPOSITION. *The power α^n of the element in cohomology determined by the subgroup H is represented by the cocycle*

$$P_n = Q_n \uparrow_H^G \oplus \cdots \oplus Q_0 \uparrow_H^G \to Q_0 \uparrow_H^G \to k$$

where the first map is projection onto the summand $Q_0 \uparrow_H^G$ and the second is the augmentation. If $\beta \in H^r(G, K)$ is represented by a cocycle which factors through $Q_r \uparrow_H^G$ in the manner $\hat{\beta} : P_r \to Q_r \uparrow_H^G \to k$ then $\beta\alpha^n$ is represented by the corresponding map $P_{n+r} \to Q_r \uparrow_H^G \to k$ defined similarly on the summand Q_r of P_{n+r}.

As an application of this we may deduce in the Q_8 example, for instance, that α and α^2 are non-zero, but $\alpha^3 = 0$ since there is a component of the differential e_3 of the form $Q_0 \uparrow_H^G \xrightarrow{j} Q_2 \uparrow_H^G \subseteq P_2$ which shows that the map $P_3 \to Q_0 \uparrow_H^G \xrightarrow{\epsilon} k$ factors through P_2, and hence is a coboundary.

Property (iii) The corestriction map from $H^*(H, k)$ is easily identified.

PROPOSITION. *Suppose that $\hat{\zeta} : Q_n \to k$ is a cocycle which represents some element ζ of $H^n(H, k)$. Then $\mathrm{cores}_H^G \zeta$ is represented by the cocycle*

$$P_n \xrightarrow{u_n} Q_n \uparrow_H^G \to k \otimes_{kG} Q_n \uparrow_H^G \cong k \otimes_{kH} Q_n \xrightarrow{\hat{\zeta}} k$$

where the map on the right, by abuse of notation, is the one induced by $\hat{\zeta}$, and the map u_n on the left is the one which occurs at degree n in the map $u : \mathcal{P} \to \mathcal{Q} \uparrow_H^G$.

As an application of this in the Q_8 example, if we denote by ζ_i a non-zero element in $H^i(\langle i \rangle, k)$ then we see that $\mathrm{cores}\, \zeta_0 = 0$ and $\mathrm{cores}\, \zeta_1 = 0$, whereas $\mathrm{cores}\, \zeta_2$ is non-zero and is represented by a cocycle defined on the summand $Q_0 \uparrow_H^G$ of P_2, and $\mathrm{cores}\, \zeta_3$ is non-zero and is represented by a cocycle defined on the summand $Q_1 \uparrow_H^G$ of P_3. This is because in our calculation of u all components of u_0 and u_1 lie in the radical of kG, whereas in u_2 and u_3 there were non-radical components in the appropriate places, and the cocycles we obtain are not coboundaries because the boundary components defined on these summands all lie in the radical.
We may deduce the following.

THEOREM. $\mathrm{cores}_H^G \zeta = 0$ *for all ζ in $H^*(H, k)$ if and only if the Hilbert-Poincaré series of the cohomology rings satisfy*

$$P_G(t) = \frac{P_H(t)}{(1-t)}.$$

This is the situation which occurs, for example, when we take $G = D_{2^n}$ to be dihedral of order 2^n, and $H = C_{2^{n-1}}$ to be the subgroup which is cyclic of order 2^{n-1}.

Property (iv) Restrictions to H are easily identified.
As a complex of kH-modules the complex $\mathcal{Q} \uparrow_H^G$ decomposes as $\mathcal{Q} \oplus g\mathcal{Q}$ where g lies in G but not in H, and using the embedding of kH-complexes $\mathcal{Q} \to \mathcal{Q} \uparrow_H^G$ which this gives we construct a map of complexes of kH-modules $\mathcal{Q} \to \mathcal{P}$ specified as the composite of the inclusions

$$\mathcal{Q} \to \mathcal{Q} \uparrow_H^G \to \mathcal{P}.$$

For the right-hand map we identify P_n as $Q_n \uparrow_H^G \oplus \cdots \oplus Q_0 \uparrow_H^G$ and include $Q_n \uparrow_H^G$ as the first summand.

PROPOSITION. *(i) The map of kH-complexes $\mathcal{Q} \to \mathcal{P}$ lifts the identity map on k:*

$$\begin{array}{ccc} \mathcal{Q} & \longrightarrow & k \\ \big\downarrow & & \| \\ \mathcal{P} & \longrightarrow & k \end{array}.$$

(ii) At the level of cocycles the restriction map to H is described by restricting to the summand $Q_n \uparrow_H^G$ of P_n and then restricting to Q_n.

With trivial coefficients and a minimal resolution $\mathcal{Q}$ the restriction is non-zero if and only if the component on $Q_n \uparrow_H^G$ is non-zero.
In the Q_8 example we see that in each of degrees 0 and 1 there is a cohomology class with non-zero restriction, but all classes in degrees 2 and 3 restrict to zero.

This description of the restriction map is useful when we compute products in $H^*(G, k)$. The components on the $Q_n \subseteq P_n$ of cocycles multiply the same way as the restrictions of these cocycles to H. Provided we know the cohomology ring of H, this helps in building up the cohomology ring of G inductively.