Pre-class Warm-up!

For a function $f : R \to R$, which (if any) of the following is a correct definition that f is continuous at x?

a. For all $e > 0$ there exists $d > 0$ so that $|f(x) - f(y)| < e$ implies $|x - y| < d$. 4

b. For all $e > 0$ there exists $d > 0$ so that $|x - y| < e$ implies $|f(x) - f(y)| < d$. d e 4 ✓

c. There exists $e > 0$ so that for all $d > 0$, $|f(x) - f(y)| < e$ implies $|x - y| < d$.

d. There exists $e > 0$ so that for all $d > 0$, $|x - y| < e$ implies $|f(x) - f(y)| < d$. 2

Which (if any) of the following informal definitions expresses the property that f is continuous?

a. Whenever x is close to y, then $f(x)$ is close to $f(y)$. ✓

b. Whenever $f(x)$ is close to $f(y)$, then x is close to y. ✗

c. f has no jumps. ✓

Why, in mathematics, do we take the definition that we do, rather than something more intuitive?

- Because we are perverse — Because - - —

How do you get to learn the definition of continuity?

- Memorization
- Figuring it out
- Other?

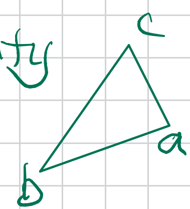
Chapter 1: Metric spaces

Distance functions.

Definition. Let A be a set. A function $d: A \times A \to \mathbb{R}$ is a distance function $\Leftrightarrow$ it satisfies (i) – (iv).

(i) d(a,b) = 0 if and only if a = b.
(ii) d(a,b) + d(a,c) ≥ d(b,c) for all a, b, c in A
(iii) d(a,b) ≥ 0 for all a, b in A
(iv) d(a,b) = d(b,a) for all a, b in A.

(ii) is called the triangle inequality



(i)′ d(a,a) = 0 for all a in A.

If d satisfies (i)′, (ii), (iii), (iv) it is called a pseudometric

A distance function is also called a metric.

Examples. 1. The `usual metric' on R^n.

$$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$$= \|x - y\|$$

2. The discrete metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & x \neq y \end{cases}$$

Proposition. This is a metric.

Proof. (i), (iii), (iv) ✓

(ii). Every number in (ii) is 0 unless some of a, b, c are different.

Case $b = c$ ✓

$b \neq c \Rightarrow (a \neq b \text{ or } a \neq c)$

so we get 2 or $1 \geq 1$ ✓

Activity: Worksheet 2, numbers 1 and 2.

(t) & (ii) $\Rightarrow$ (iii)

$d(a,b) + d(a,b) \geq d(b,b)$ on taking $b=c$,

$2d(a,b) \geq 0$ using only (i)'.

(iv) $d(a,b) + d(a,a) \geq d(b,a)$ (taking $a=c$)

$d(b,a) + d(b,b) \geq d(a,b)$

Using (i)' $d(a,b) \geq d(b,a)$.

Similarly $d(b,a) \geq d(a,b)$.

	Not difficult Yes	No
a	50	50
b	Most	
c	Most	
d		Most

Continuity

(A, d_A)

Definition. Let (A,d_A), (B,d_B) be metric spaces.
A function f : A -> B is continuous at x in A if

$\forall \epsilon > 0 \ \exists \delta > 0$ so that

$d_A(x,y) < \delta$ implies $d_B(f(x), f(y)) < \epsilon$.

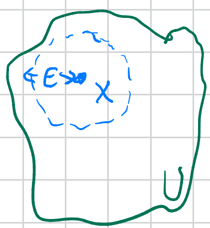
We say f is continuous $\Leftrightarrow$ f is continuous at every x in A.

Definition. Given e > 0 the open ball B_e(x)

$$B_\epsilon(x) = \{ y \in A \mid d_A(x,y) < \epsilon \}$$

A subset U of A is open if

$\forall \ x \in U \ \exists \ \epsilon > 0$ so that $B_\epsilon(x) \subseteq U$



Worksheet 2 qn. 7 Let R ($\mathbb{R}$) have the usual metric,
let (A,d) have the discrete metric. Show
a. All functions f : A -> R are continuous,
b. There is no injective continuous f : R -> A.

Proof a. Let $x \in A$ and let $\epsilon > 0$. We find $\delta > 0$ so that ... Take $\delta = \frac{1}{2}$. then if $d_A(x,y) < \frac{1}{2}$ it follows that $x = y$ so $d(f(x), f(y)) = 0 < \epsilon$. The condition for continuity at x is satisfied.

b. Let $f : \mathbb{R} \to A$ be continuous, so the condition for continuity holds. Take $\epsilon = \frac{1}{2}$. Then $\forall x \in A \ \exists \delta$ so that if $d_R(x,y) < \delta$ then $d_A(f(x), f(y)) < \frac{1}{2}$ so $f(x) = f(y)$.

It follows that $f^{-1}(z)$ is open for every $z \in A$ and $\mathbb{R} = \bigsqcup_{z \in A} f^{-1}(z)$ is a disjoint union of open sets. This cannot be unless there is only one set, so f is not injective. Why cannot this be? See next page

This is correct, but a shorter conclusion was suggested.

To show $\mathbb{R} = \bigsqcup_{z \in A} f^{-1}(z)$ is impossible unless f takes only one value.

Fix $x \in \mathbb{R}$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(y) = d_A(f(x), f(y))$

Claim: g is continuous.

Check: given $\epsilon > 0$ we can find ~~δ so that~~

Take δ so that $B_\delta(x) \subseteq f^{-1}(f(x))$

Now $d_{\mathbb{R}}(y, x) < \delta \Rightarrow f(x) = f(y)$

so $d_A(f(x), f(y)) = 0 < \epsilon$.

If f takes more than one value then $g(x) = 0$ and $g(y) = 1$, for some other y. By the intermediate value theorem $\exists\, z \in \mathbb{R}$ with $g(z) = \frac{1}{2}$. This is not possible.

Hence f takes only one value and is not injective. $\square$

The above is correct, but proves something stronger than what is needed.

Because there exists $d>0$ so that $|x - y| < d$ implies $f(x) = f(y)$, this immediately implies that f is not injective. This is sufficient to do this problem.