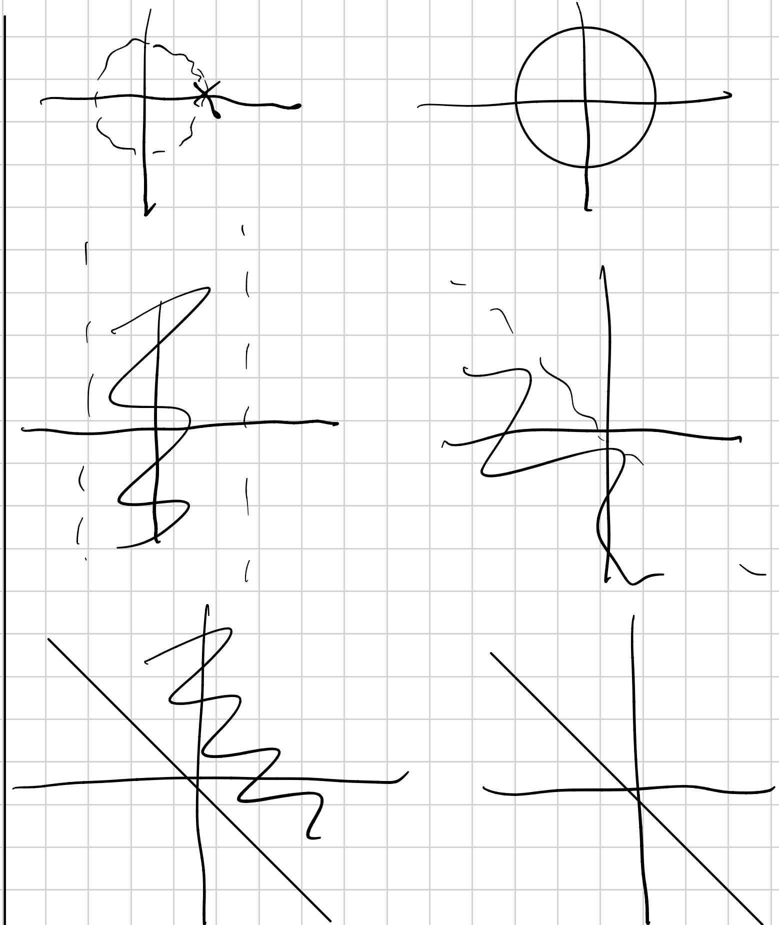
Pre-class Warm-up!!!

Worksheet ' 2 Question 10:
Which of the following subsets of R^2 (with the usual topology) are open?

	Yes	No
a. $\{(x,y) \mid x^2+y^2<1\} \cup \{(1,0)\}$		✓
b. $\{(x,y) \mid x^2+y^2 \leq 1\}$		✓
c. $\{(x,y) \mid \lvert x\rvert < 1\}$	✓	
d. $\{(x,y) \mid x+y<0\}$	✓	
e. $\{(x,y) \mid x+y \geq 0\}$		✓
f. $\{(x,y) \mid x+y=0\}$		✓



Worksheet 2 qn 8, 10, 11b

A slightly different example.

Let $X = \{0, 1/n \mid n \text{ is a positive integer}\}$ as a subset of $\mathbb{R}$ with the same distance function as $\mathbb{R}$.
What are the open subsets of X ?

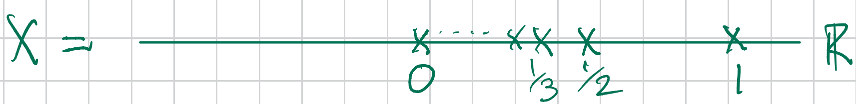
a. All subsets of X.

b. All finite subsets of X, together with X.

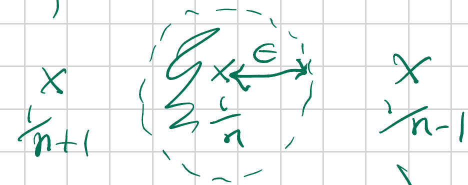
c. All subsets of X, with the proviso that if 0 is in the set then for some N, every $1/n$ is in the set for all $n \geq N$. 3

d. All subsets of X, with the proviso that if 0 is in the set then the set is infinite. 2

e. None of the above. several



Every set $\{\frac{1}{n}\}$, $n \in \mathbb{N}$ $n \geq 1$ is open



$\epsilon < \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)$

shows a ball in X whose only point is $\{\frac{1}{n}\}$.

The balls containing 0 have the form $\{0, \frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \ldots\}$

Every open set is a union of such balls.

Such open sets are described by c.

Theorem 1.9 A function f : M_1 -> M_2 between two metric spaces is continuous if and only if for all open sets U in M_2 the set f^{-1}(U) is open in M_1.

Can we recall the definitions of continuous and open?

Proof: "$\Rightarrow$" Suppose f is continuous. Let $U \subseteq M_2$ be an open set. We show that $f^{-1}(U)$ is open in M_1. Let $x \in f^{-1}(U)$. Because U is open and $f(x) \in U$, $\exists \epsilon > 0$ so that $B_\epsilon(f(x)) \subseteq U$. Because f is continuous, $\exists\ \delta > 0$ so that $d_1(x,y) < \delta$ implies $d_2(f(x), f(y)) < \epsilon$

i.e. $f(y) \in B_\epsilon(f(x)) \subseteq U$. Thus $B_\delta(x) \subseteq f^{-1}(U)$

Thus $f^{-1}(U)$ is open.

"$\Leftarrow$" Suppose $\forall$ open U, $f^{-1}(U)$ is open. We verify $\forall x, \forall \epsilon > 0\ \exists\ \delta > 0$ so that $d(x,y) < \delta$ implies $d_2(f(x), f(y)) < \epsilon$

Suppose we are given x and $\epsilon > 0$.

Because $B_\epsilon(f(x))$ is open in M_2, $f^{-1}(B_\epsilon(f(x)))$ is open in M_1 and x lies in it. Thus $\exists\ \delta > 0$ so that $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$. Thus $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$. This is the same as $d_1(x,y) < \delta \Rightarrow d_2(f(x), f(y)) < \epsilon$.

$\square$

Not in the book.

Definition: Two metrics on A are equivalent $\Leftrightarrow$ they define the same open sets

Application. We revisit:

Worksheet 2 qn. 7 Let R have the usual metric, let (A,d) have the discrete metric. Show

a. All functions f : A -> R are continuous,

b. There is no injective continuous f : R -> A.

Solution a. f is continuous

$\iff f^{-1}(U)$ is open $\forall$ open $U \subseteq \mathbb{R}$

If A has the discrete metric then every subset of A is open so $f^{-1}(U)$ will always be open. $\square$

b. Let $f : \mathbb{R} \to A$ be continuous.

In A each point $\{a\}$ is an open set, so $f^{-1}(\{a\})$ is open in $\mathbb{R}$.

If $f^{-1}(\{a\})$ is non-empty it contains an open interval, so is infinite.

All these points are mapped to a so f is not injective. $\square$

Chapter 2: Topological spaces

Definition 2.1. Let X be a set and $\mathcal{U}$ a collection of subsets of X. We say X (or $(X, \mathcal{U})$) is a topological space if

(i) $\emptyset, X \in \mathcal{U}$

(ii) Arbitrary unions of sets in $\mathcal{U}$ belong to $\mathcal{U}$

(iii) Finite intersections of sets $\mathcal{U}$ belong to $\mathcal{U}$

$\mathcal{U}$ is called the topology on X.

Examples.

1. Metrizable topological spaces. If (X, d) is a metric space then the open susets of X form a topology.

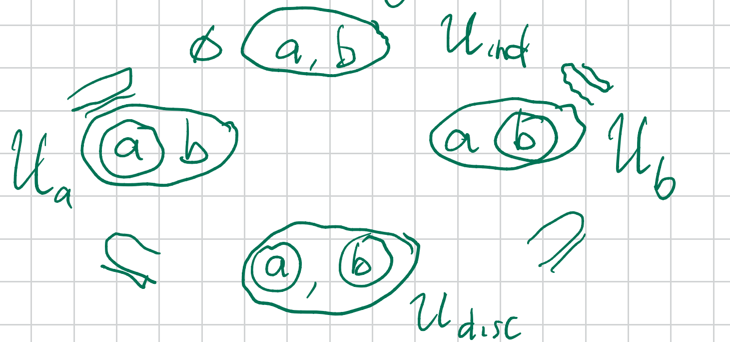
Example. The discrete topology has $\mathcal{U}$ = all subsets of X, and arises from the discrete metric.

2. The indiscrete topology.

This has $\mathcal{U} = \{\emptyset, X\}$

3. Topologies on 2 points.

Let $X = \{a, b\}$



4. The finite complement topology.

We take $\mathcal{U}$ = subsets A of X so that $X - A$ is finite together with $\emptyset$

Check $X, \emptyset \in \mathcal{U}$ ✓

arbitrary unions ... ✓

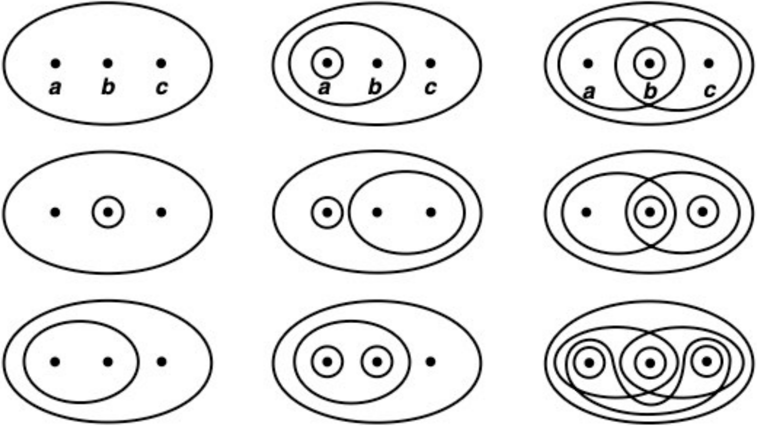
finite intersections ... ✓

The Zariski topology on Spec $\mathbb{Z}$

= { prime ideals of $\mathbb{Z}$ }

= { {0}, $p\mathbb{Z}$ where p is prime }

The open sets are the complements of sets of primes that contain some given integer.



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