Pre-class Warm-up !!

Is the following a topology on the real numbers R ?

U = the collection of all finite subsets of R, together with R itself.

a. Yes
b. No ✓ arbitrary unions of finite subsets need not be finite.

Let U, V be subsets of X.
Is it obvious that $(X-U) \cap (X-V) = X - (U \cup V)$?

a. Yes b. No

Is it obvious that $(X-U) \cup V = X - (U \cap V)$?

a. Yes b. No

What can we take as a working definition of 'obvious'?

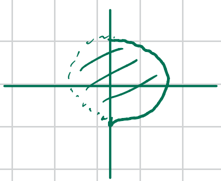
Some definitions

Interior, closed, closure, boundary, (limit points)

Definition. Let Y be a subset of a topological space X.
The interior $\mathring{Y} = Y^\circ = \mathrm{Int}(Y)$ is the largest open subset of Y.
It is the union of all open subsets of X contained in Y.

$$Y^\circ = \bigcup_{\substack{A \subseteq Y \\ A \text{ open in } X}} A$$

Example. $Y = \{(x,y) \mid x^2+y^2<1\}$
$\cup \{(x,y) \mid x^2+y^2=1,\ x \geq 0\}$



$$Y^\circ = \{(x,y) \mid x^2+y^2<1\}$$

Definition. Y is closed $\iff$ $X - Y$ is open

Example $[a,b] \subseteq \mathbb{R}$ is closed because $\mathbb{R} - [a,b] = (-\infty, a) \cup (b, \infty)$ is open.

The closure of Y is the smallest closed set that contains Y

$$\overline{Y} = \bigcap_{\substack{A \supseteq Y \\ A \text{ is closed in } X}} A$$

Note $X - \overline{Y} = X - (\bigcap A) = \bigcup (X - A)$ which is open, so $\overline{Y}$ is closed.

Definition. The boundary of Y is

$$\partial Y = \overline{Y} - \mathring{Y}$$

Example on left. $\overline{Y} = \{(x,y) \mid x^2+y^2 \leq 1\}$

$$\partial Y = \{(x,y) \mid x^2+y^2 = 1\}.$$

Recall: A slightly different example.

Let $X = \{0, 1/n \mid n$ is a positive integer$\}$ as a subset of $\mathbb{R}$ with the same distance function as $\mathbb{R}$.

What are the open subsets of X ?

The correct answer was:

c. All subsets of X, with the proviso that if 0 is in the set then for some N, every $1/n$ is in the set for all $n \geq N$.

What is the interior of

a. $Y = \{1\}$ $\quad Y^\circ = Y$

b. $Z = \{0\}$ $\quad Z^\circ = \emptyset$

c. $W = \{1/n \mid n \geq 1\}$ $\quad W^\circ = W$

Answers:

a. empty set

b. X

c. Y

d. Z

e. W

Same question with closures.

$\overline{Y} = Y$ b/c $X - Y = \{0, \frac{1}{n} \mid n \geq 2\}$ is open, so Y is closed

$\overline{Z} = Z$ b/c $X - Z$ is open (doesn't contain 0)

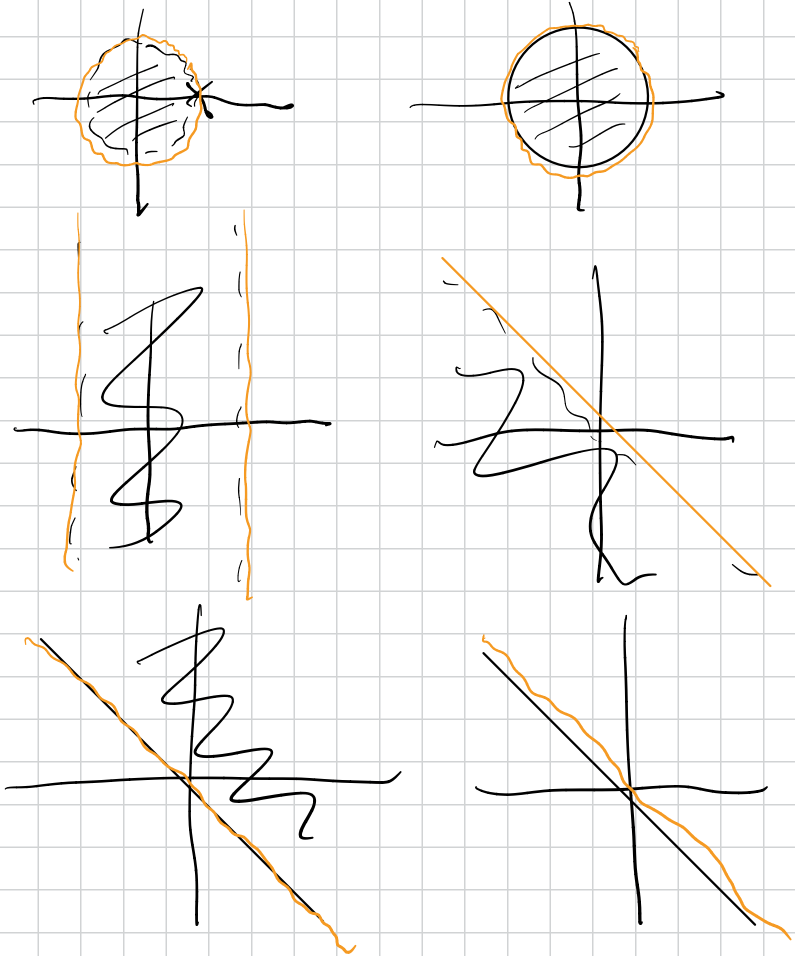
$\overline{W} = X$ (W itself is not closed)

Recall:

Pre-class Warm-up!!!

Worksheet ' 2 Question 10:
Which of the following subsets of R^2 (with the usual topology) are open? closed?

	Yes	No
a. $\{(x,y) \mid x^2+y^2<1\} \cup \{(1,0)\}$		✓
b. $\{(x,y) \mid x^2+y^2 \leq 1\}$	✓	
c. $\{(x,y) \mid \lvert x\rvert < 1\}$		✓
d. $\{(x,y) \mid x+y<0\}$		✓
e. $\{(x,y) \mid x+y \geq 0\}$	✓	
f. $\{(x,y) \mid x+y=0\}$	✓	



Worksheet 3 questions 3, 4, 5

Worksheet 3

3. Obvious?

Theorem ... $\mathcal{U} = \{X - V\}$ is a topology

Yes No

4. Obvious?

Theorem. In a discrete space every subset is open and closed

Yes No

5. Obvious?

Theorem. If X has only finitely many points, each of which is closed, then it has the discrete topology.

Yes No

The last implication of part c that LHS is contained in RHS was not done in class. It proceeds similarly and is an exercise.

A result to make you think about the definitions

Proposition. Let Y be a subset of a topological space X.
a. The interior
Int(Y) = { x in X | there exists an open set U contained in Y, with x in U. }
b. The closure
Y^– = { x in X | for all open U with x in U, $U \cap Y \neq \emptyset$ }.
c. The boundary
∂Y = { x in X | for all open U with x in U, $U \cap Y \neq \emptyset$ and $U \cap (X-Y) \neq \emptyset$ }

Which definition of the closure do you prefer?
a. The initial definition
b. The property given here

Proof. a. $\mathrm{Int}(Y) = \bigcup$ open sets $\subseteq Y$.
"RHS $\subseteq$ LHS" If $x \in$ RHS then $\exists\, x \in U \subseteq Y$ with U open. Thus $x \in U \subseteq \mathrm{Int}(Y)$.
"LHS $\subseteq$ RHS". Let $x \in \mathrm{Int}(Y)$. Take $U = \mathrm{Int}(Y)$ to show $x \in$ RHS.

b. $\bar{Y} = \bigcap$ closed $A \supseteq Y$.
RHS $\subseteq$ LHS. Let $x \in$ RHS; we show $x \in$ every closed set A containing Y. If not $\exists$ closed A with $x \notin A$, $A \supseteq Y$. Thus $x \in X - A$, which is open so $(X-A) \cap Y \neq \emptyset$

This implies $A \not\supseteq Y$, a contradiction.
LHS $\subseteq$ RHS Let $x \in$ LHS $= \bar{Y}$ so $x \in$ every closed set containing Y. Let $x \in U$, U open. We show $U \cap Y \neq \emptyset$ by assuming otherwise; $U \cap Y = \emptyset$ so $X - U$ is closed and contains Y. Thus $x \in X - U$. This contradicts $x \in U$.
(c) RHS $\subseteq$ LHS. Suppose $\forall$ open U, $x \in U$ we have $U \cap Y \neq \emptyset$ and $U \cap (X-Y) \neq \emptyset$. Then $x \in \bar{Y}$ by (b). We show $x \notin Y^{\circ}$ by showing $x \in X - Y^{\circ} = \bigcap$ closed sets containing $X-Y$ $= \overline{X-Y}$. This follows from (b) applied to $X-Y$.