Pre-class Warm-up!!!

Let Y be a subset of a topological space X. Which do you like better as the definition of the closure of Y ?

a. The smallest closed set containing Y. 4

b. The intersection of the closed sets containing Y. 2

c. The set of points x in X for which, for every open set U containing x , U contains a point of Y. 1

d. The set of points x in X for which, for every open set U containing x , U contains a point of Y and U contains a point of X – Y. 0

e. Y together with all its boundary points 4

point in the boundary ∂Y

$\|$

boundary point in Marsden & Tromba and Colley.

The definition for Y to be closed given by Marsden and Tromba is that Y contains all its boundary points.

Some more definitions of terms you need to know

Neighborhood, basis for a topology

not the same as in calculus texts.

not in Kosniowski's book

Definition. Let $x \in X$, a topological space.

A neighborhood of x is a set $N \subseteq X$ so that $x \in N$ and there is an open set U with $x \in U \subseteq N$.

For example: In $\mathbb{R}$, $[0,1]$ is a neighborhood of $\frac{1}{2}$

Marsden & Tromba define what I call an open neighborhood.

Definition. Let X be a set. A basis for a topology on X is a collection $\mathcal{B}$ of subsets of X so that

(i) X is the union of the sets in $\mathcal{B}$

(ii) if x lies in the intersection $B_1 \cap B_2$ of two elements of $\mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ so that

$$x \in B_3 \subseteq B_1 \cap B_2$$

Example

In $\mathbb{R}^n$ the balls $B_\epsilon(x)$, $\epsilon > 0$, $x \in X$ is a basis for a topology on X.

Definition. The topology generated by a basis is $\mathcal{U} :=$ all unions of sets in $\mathcal{B}$.

Proposition. This $\mathcal{U}$ is a topology on X.

Proof. Check (i) $\emptyset$, $X \in \mathcal{U}$?? Uh-Oh!! the empty intersection

(ii) arbitrary unions

(iii) finite intersections.

We say the basis is a basis for a given topology if the given topology is generated by the basis.

Examples In a metric space

$\mathcal{B} = \{B_\epsilon(x) \mid \epsilon > 0, x \in X\}$ is a basis for the metric topology.

Definition. A subbasis for a topology on X is a collection S of sets whose union is X.
The topology it generates has as its open sets the unions of finite intersections of sets in S.

It is the smallest topology for which $\mathcal{U} = \{\text{open sets}\} \supseteq S$.

Worksheet 3: 1, 2e, 8fg ?

True / false? :

a. the interior of the closure equals the interior.

b. the closure of the interior equals the closure.

Question. Let f : A -> B be a mapping, W a subset of A and Z a subset of B.

a. Is it obvious that necessarily
f^{-1} (B – Z) = A – f^{-1}(Z)
50 50
Yes / No

If $x \in A$ then either $f(x) \in Z$ or $f(x) \notin Z$, so $x \in f^{-1}(Z)$ or $x \in f^{-1}(B-Z)$ and not both.

$A = f^{-1}(Z) \sqcup f^{-1}(B-Z)$ is a disjoint union so $A - f^{-1}(Z) = f^{-1}(B-Z)$.

b. Is it obvious that necessarily
f(A – W) = B - f(W) ?
Yes / No

Definition. A function f : X -> Y between topological spaces is continuous if $f^{-1}(U)$ is open for every open set $U \subseteq Y$.

Examples

1. Usual continuous functions between spaces like $\mathbb{R}^n$

2. The inclusion map $\{0, \frac{1}{n} | n \geq 1\} \to \mathbb{R}$ is continuous.

3. Let A have the discrete topology
Let B have the indiscrete topology = the concrete topology.

Every map $A \to C$ (some other space) is continuous
Every map $C \to B$ is continuous.

If $A = B$ as sets then $A \xrightarrow{1} B$ is continuous but $B \xrightarrow{1} A$ is not continuous if $A = B$ has ≥ 2 points.

The notions 'bijective' and 'has an inverse' are different here.

Theorems.

3.3. f : X -> Y is continuous <=> for every closed set $C \subseteq Y$, $f^{-1}(C)$ is closed in X.

3.5 The composite of continuous functions is continuous.

Proof. f is continuous

$\iff f^{-1}(\text{open } U)$ is open

$\iff f^{-1}(Y - U) = X - (\text{open})$, which is closed.

$\iff f^{-1}(\text{closed } Y)$ is closed. $\square$

Proof of 3.5 Let $f : X \to Y$, $g : Y \to Z$ be continuous.

We show $gf : X \to Z$ is continuous.

Let $U \subseteq Z$ be open. Then $g^{-1}(Z) \subset Y$ is open. Hence $f^{-1}(g^{-1}(Z^{U}))$

$= (gf)^{-1}(Z)$ is open in X. $\square$

Question: Which do you like better:

a. the proof of 3.5, or

b. the usual proof using the epsilon-delta definitions.