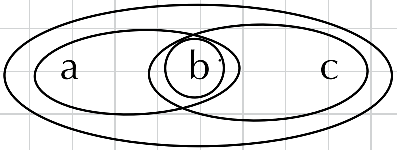
Pre-class Warm-up!!!

Let X = {a,b,c} be the topological space with open sets {ø, {b}, {a,b}, {b,c}, {a,b,c} }



Let f, g : X -> X be the mappings
f(a)=a, f(b)=a, f(c)=b
g(a)=a, g(b)=a, g(c)=c

1 Is f continuous? Yes / No

$f^{-1}(\{b\}) = \{c\}$ is not open

2 Is g continuous? Yes / No

$g^{-1}(\{b,c\}) = \{c\}$ is not open.

3 How many closed sets does X have?

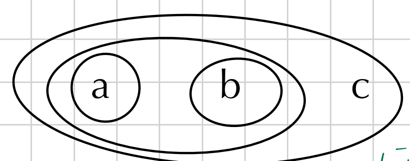
a. 2
b. 3
c. 4
d. 5 ✓
e. None of the above

open sets & closed sets are in bijection.

If $f : X \to Y$, $V \subseteq Y$ then

$f^{-1}(V) := \{x \in X \mid f(x) \in V\}$

Another space



not continuous $h:$ $a \to a$, $b \to b$, $c \to b$

continuous $j:$ $a \to b$, $b \to b$, $c \to c$

Open and closed mappings $\simeq$ maps $=$ functions

Definition. A map f : X -> Y is open <=> for all open sets $U \subseteq X$, $f(U)$ is open.

It is closed <=> for all closed sets $V \subseteq X$, $f(V)$ is closed.

Example 1. $f : (0,1) \longrightarrow \mathbb{R}$, the inclusion mapping.

f is open, but not closed. If $U \subseteq (0,1)$ is open then $\forall\, x \in (0,1)\ \exists\, \epsilon > 0$, $B_\epsilon(x) \subseteq (0,1)$ so U is also open in $\mathbb{R}$ using the same $B_\epsilon(x)$. So f is open.
On the other hand $V = (0, \frac{1}{2}]$ is closed in $(0,1)$ because $(0,1) - V = (\frac{1}{2}, 1)$ is open in $(0,1)$.

$f(V) = (0, \frac{1}{2}]$ is not closed in $\mathbb{R}$.

2. $f : [0,1] \longrightarrow \mathbb{R}$ (inclusion) is closed but not open.

3. $f : (-10, 10) \longrightarrow \mathbb{R}$, $f(x) = x^2$,

Is this open? Yes / No
closed? Yes / No

Definition. ' f : X -> Y is a homeomorphism
<=> f is continuous and there is a continuous map $g : Y \to X$ so that $fg = 1_Y$ and $gf = 1_X$.
i.e. f has an inverse among continuous maps.
<=> $f \simeq$ "it"
(i) it is bijective,
(ii) it is continuous,
(iii) f^{-1} is continuous. or f is open

Book's examples: $f = 1_{\mathbb{R}} : (\mathbb{R}, d_0) \to (\mathbb{R}, d_1)$ is a homeomorphism.

Other examples $(0,1)$ is homeomorphic to $\mathbb{R}$.
Write $X \cong Y$ to mean they are homeomorphic.

Posets - on Worksheet 4

A partially ordered set P (or poset) is a set with a transitive, reflexive and skew-symmetric relation $x \leq y$, so that

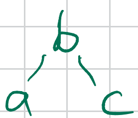
(i) $x \leq x$ for all x in P

(ii) $x \leq y$ and $y \leq z$ implies $x \leq z$

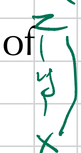
(iii) $x \leq y$ and $y \leq x$ implies $x = y$

If only conditions (i) and (ii) are satisfied, it is called a pre-ordered set.

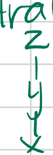
For example $\{a, b, c\}$ $a \leq b$ $c \leq b$ $a \leq a$ $b \leq b$ $c \leq c$ describes such a relation. $\{\text{cherry}, \text{orange}, \text{pecan}\}$ (a, b, c)



This is the Hasse diagram. Instead of



draw

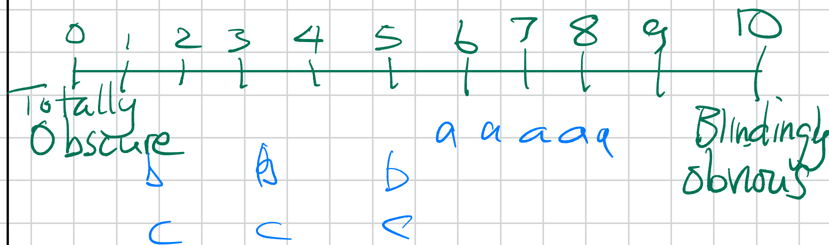


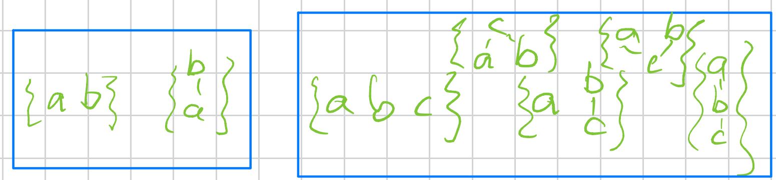
À Given a poset P, let U be the collection of subsets U of P satisfying x in P and $y \leq x$ implies y in P

Show that U is a topology on P. There are other ways to make P into a topological space.

Work on questions 1 and 2 on Worksheet 4

The indiscrete topology on X has open sets $\{\emptyset, X\}$ only.





Chapter 4: The induced topology, or subspace topology, or relative topology

Definition Let X be a topological space and $S \subseteq X$ a subset. We define a collection of subsets of S

$$\mathcal{U}_S = \{ S \cap W \mid W \text{ is an open subset of } X \}$$

Is it obvious that this is a topology?

Check: ~ $\emptyset, S \in \mathcal{U}_S$
~ arbitrary unions lie in $\mathcal{U}_S$
~ finite intersections lie in $\mathcal{U}_S$.

What about the closed sets?

They are the intersections of S with closed subsets of X.

A closed subset of S has the form $S - (S \cap W)$, W open in X.

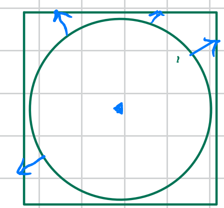
This equals $S \cap (X - W) = S \cap$ closed.

Example: $[0,1]$ as a subset of $\mathbb{R}$. The open subsets of $[0,1]$ are $W \cap [0,1]$, $W \subseteq \mathbb{R}$ is open.

Thus $(-1, \frac{1}{2}) \cap [0,1] = [0, \frac{1}{2})$ is open in $[0,1]$ with subspace topology.

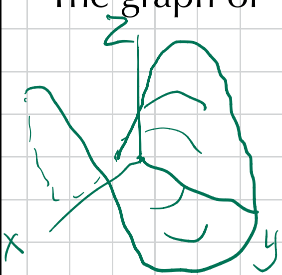
We get the same open sets as if we take $[0,1]$ as a metric space.

Example: the circle and the square are homeomorphic



This Map from the circle to the square is a homeomorphism

The graph of f(x,y) = x^2 - y^2 in $\mathbb{R}^3$



is homeomorphic to $\mathbb{R}^2$ via

$$(x,y) \longmapsto (x, y, f(x,y))$$