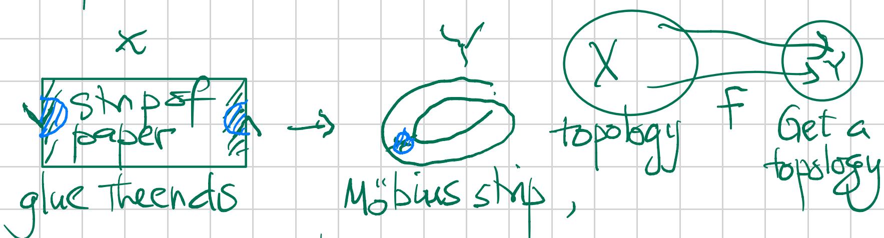
Chapter 5: Quotient topology (and groups acting on spaces)

Definition 5.1. Let f : X -> Y be a surjective mapping from a topological space X to a set Y. The quotient topology on Y (with respect to f) is the family

$\mathcal{U}_f$ = { U | f^{-1}(U) is open in X }



open set in X = f^{-1}(set shown in Y)

We put a topology on the Möbius strip.

Proposition. (a) If Y has the quotient topology then f is continuous.
(b) The quotient topology is the ~~smallest~~ largest topology on Y so that f is continuous. ? ? ?

Proof (a) We check ∀ open $U \subseteq Y$, $f^{-1}(U)$ is open to verify that f is continuous. This holds from the definition of open sets in Y.

b). in any topology on Y for which f is continuous, ∀ open $U \subseteq Y$, $f^{-1}(U)$ must be open, so this topology must ~~contain~~ be contained in the quotient topology. □

True or false? Let S be a subset of a topological space X.

(i) The induced topology on S is the largest topology on S so that the inclusion map g : S -> X is continuous. T / F ?

(ii) The induced topology on S is the smallest topology on S so that the inclusion map g : S -> X is continuous. T / F ?

Pre-class Warm-up!!!

Can you remember what the ingredients in the quotient topology are?
We have a map surjective f : X -> Y. Which (if any) is a correct description?

a. X is a topological space and U is open in X <=> f(U) is open in Y.

b. Y is a topological space and U is open in X <=> f(U) is open in Y.

c. X is a topological space and f^{-1}(U) is open in X <=> U is open in Y. ✓

d. Y is a topological space and f^{-1}(U) is open in X <=> U is open in Y.

Second question: how many ways can you think of to define the n-sphere S^n ?

a. 0
b. 1
c. 2
d. ≥ 3

$$S^n = \{ v \in \mathbb{R}^{n+1} \mid \|v\| = 1 \}$$

Exam on Wednesday 10/1/2025.

50 minutes. Chapters 1-4 (induced topology)

You can use books, notes, worksheets, quizzes.

Examples for the quotient topology

Example: projective space.

Questions:

1. Have you ever encountered a theorem in projective geometry before? Yes / No
2. How many ways have you seen before of constructing a projective space, such as the real projective plane RP^2 ?

a. 0
b. 1
c. 2
d. 3
e. >3

Pairs of antipodal points

We start with $S^n = \{ v \in \mathbb{R}^{n+1} \mid \|v\| = 1 \}$.

Define $\mathbb{RP}^n := \{ \{v, -v\} \mid v \in S^n \}$

We have a surjective map $\pi : S^n \to \mathbb{RP}^n$

$v \longmapsto \{v, -v\}$

We give $\mathbb{RP}^n$ the quotient topology.

Lines in R^{n+1} (not done in the book)

$\mathbb{RP}^n$ is the set of 1-dimension subspaces $\{ \lambda v \mid \lambda \in \mathbb{R} \}$ for some $v \in \mathbb{R}^{n+1}$.

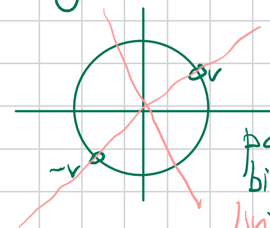
We have a surjective map $\mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n$

$v \longmapsto \{ \lambda v \mid \lambda \in \mathbb{R} \}$ if $v \neq 0$.

$\mathbb{R}^{n+1} - \{0\}$ has the usual metric topology.

We give ~~give~~ $\mathbb{RP}^n$ the quotient topology.

In 2D:



we get the projective line, pairs of antipodal points biject with lines through 0.

$\mathbb{RP}^1$ is a circle.

A quotient of a ball (a disk in 2-D)

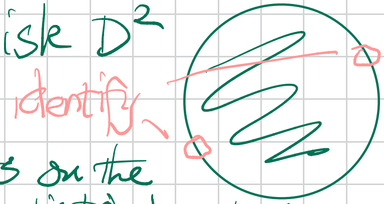
Let $N \subseteq S^n$ be the northern hemisphere $N = \{ v \mid \|v\| = 1, v_{n+1} \geq 0 \}$

In $\mathbb{R}^3$

We have an inclusion $N \subseteq S^n$.

The composite $N \hookrightarrow S^n \to \mathbb{RP}^n$ is surjective. $\mathbb{RP}^n$ has the quotient topology.

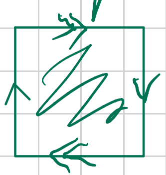
When $n=2$, $N \cong$ disk D^2



Under the map $N \longrightarrow \mathbb{R}P^2$ points on the interior of D^2 go to distinct pairs. Points on ∂D^2 go to the same pair if they are antipodal.

A quotient of a square

A square is homeomorphic to D^2.



We get $\mathbb{R}P^2$ by glueing ∂(square) to itself according to the scheme shown.

This is a manageable picture of $\mathbb{R}P^2$.

Quotients given by equivalence relations

If we have an equivalence relation $\sim$ on X we can let Y be the set of equivalence classes, and we get a surjective map

$X \longrightarrow Y$, $x \longmapsto$ class of x.

We give Y the quotient topology.

Example $X = D^2 = \{(x,y) \mid x^2+y^2 \le 1\}$

Put $v \sim v$ always. If $\|v\| = 1$ put $v \sim -v$. This defines an equivalence relation.

$\mathbb{R}P^2 = D^2/\sim$

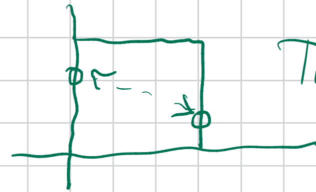
Examples for the quotient topology

The Möbius strip or band again

Ta

Take $X = \{(x,y) \mid 0 \le x, y \le 1\}$

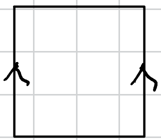
Define $v \sim v$ always and $(0,y) \sim (1,1-y) \sim (0,y)$ if $0 \le y \le 1$.



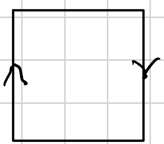
Then $X/\sim$ is the Möbius strip.

Examples for the quotient topology

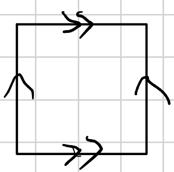
Consider the following spaces obtained by identifying points on the boundary of a square. How many can you give a name to?



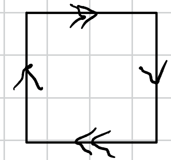
cylinder



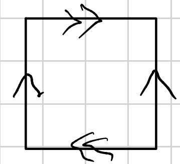
Möbius strip.



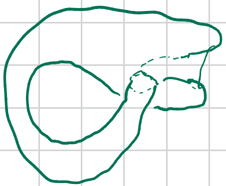
torus

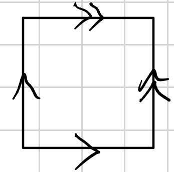


projective plane



Klein bottle.





sphere S^2

Worksheet 3 8d: Prove

$\overline{A \cup B} = \bar{A} \cup \bar{B}$ Go to the definitions.

Look at a particular example: you may get ideas!

Proof. We show LHS $\subseteq$ RHS and LHS $\supseteq$ RHS

LHS $\subseteq$ RHS: $\bar{A}$ and $\bar{B}$ are closed sets containing A and B, so $\bar{A} \cup \bar{B}$ is a closed set containing $A \cup B$. Therefore $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$ because $\overline{A \cup B}$ is the such closed set.

LHS $\supseteq$ RHS: $\overline{A \cup B}$ is a closed set containing A, so $\overline{A \cup B} \supseteq \bar{A}$. Similarly $\overline{A \cup B} \supseteq \bar{B}$.

Therefore $\overline{A \cup B} \supseteq \bar{A} \cup \bar{B}$.

Worksheet 3 8e $X - Y^{\circ} = \overline{X - Y}$.

Proof. $X - Y^{\circ} = X - \bigcup(\text{open sets} \subseteq Y)$

$= \bigcap_{\substack{\text{open sets}\\ U \subseteq Y}} (X - U) = \bigcap_{\substack{X-U\\ \supseteq X-Y}} (X - U)$

$= \bigcap_{\substack{C \supseteq X-Y\\ C \text{ is closed}}} C = \overline{X - Y}$.

Worksheet 3 8h $\partial Y = \emptyset \iff Y$ is both open and closed

Proof. $\partial Y = \bar{Y} - Y^{\circ}$ and $Y^{\circ} \subseteq Y \subseteq \bar{Y}$ so $\partial Y = \emptyset \iff Y^{\circ} = Y = \bar{Y}$

$\iff Y$ is both open and closed.

Worksheet 3 8a If $Y \subseteq F \subseteq X$ and F is closed then $\overline{Y} \subseteq F$.

Proof. $\overline{Y} = \bigcap_{C \supseteq Y} \text{closed } C$

F is among the sets in this intersection so $F \supseteq \overline{Y}$. $\square$

8c $\overline{\overline{Y}} = \overline{Y}$

Proof. $\overline{Y}$ is closed so $\overline{\overline{Y}} = \overline{Y}$ by 8b

Worksheet 3 8b Y is closed $\iff Y = \overline{Y}$.

8f. $\overline{Y} = Y \cup \partial Y$.

Proof. $\partial Y = \overline{Y} - Y^{o}$ so $\overline{Y} = \partial Y \cup Y^{o}$.
Also $Y^{o} \subseteq Y \subseteq \overline{Y}$, so
$\partial Y \cup Y^{o} \subseteq \partial Y \cup Y \subseteq \overline{Y}$ so $\partial Y \cup Y = \overline{Y}$.