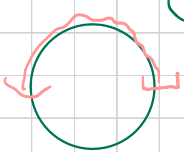
Pre-class Warm-up!!!

We can construct the circle S^1 as [0,1]/~ where ~ is the equivalence relation on the unit interval [0,1] given by x ~ y <=> x-y is an integer.
Is the quotient map [0,1] -> [0,1]/~

✗ a. open
✓ b. closed
c. neither

a. The image of $[0,\frac{1}{2})$ in S^1 is not open



but $[0,\frac{1}{2})$ is open in $[0,1]$

b. Take a closed subset V of $[0,1]$ and let $f:[0,1]\to[0,1]/\sim$ be the quotient map. Is $f(V)$ closed?

It is closed $\iff f^{-1}f(V)$ is closed.

$$f^{-1}f(V) = \begin{cases} V & \text{if } 0,1 \notin V \\ V \cup \{0\} \cup \{1\} & \text{if } 0 \in V \text{ or } 1 \in V. \end{cases}$$

both possibilities are closed in $[0,1]$.

Thus f is closed.

How to prove that different descriptions by quotient topologies give homeomorphic spaces

Theorem 5.2 Suppose we have topological spaces and mappings of sets

$X \xrightarrow{f} Y \xrightarrow{g} Z$

where f is surjective, Y has the quotient topology, and g is some mapping of sets.

Then g is continuous <=> gf is continuous.

Proof. "$\Rightarrow$" is easy. If g is continuous, we know f is continuous, so gf is continuous.

"$\Leftarrow$" Suppose gf is continuous. To show g is continuous, let $V \subseteq Z$ be an open subset.

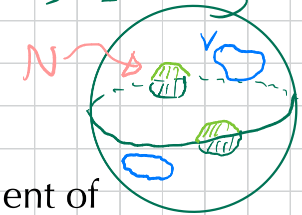
Then $g^{-1}(V)$ is open $\Leftrightarrow$ $f^{-1}(g^{-1}(V))$ is open (defn of topology on Y)

But gf is continuous so $f^{-1}(g^{-1}(V)) = gf^{-1}(V)$ is open. Thus $g^{-1}(V)$ is open, g is continuous. $\square$

Theorem.
Let N = northern hemisphere of S^2.
RP^2 constructed as a quotient of S^2 is homeomorphic to RP^2 constructed as a quotient of N.

RP^2 is constructed as the set
Pairs = { {v,-v} | v in S^2}
with topology coming as a quotient of either S^2 or N. Why are these topologies the same?

Theorem.
Let N = northern hemisphere of S^2. RP^2 constructed as a quotient of S^2 is homeomorphic to RP^2 constructed as a quotient of N. $= \{ v \in \mathbb{R}^3 \mid \|v\| = 1,\ v_3 \geq 0 \}$



RP^2 is constructed as the set
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We construct $\mathbb{RP}^2$ as a quotient space in 2 ways

$$\begin{array}{ccc} S^2 & \xrightarrow{\phi} & \text{Pairs} \\ \text{inclusion } i \ \updownarrow & & id = f \downarrow \ \uparrow g = id \\ N & \xrightarrow{\theta} & \text{Pairs} \end{array}$$

Let $\text{Pairs} = \{ \{v, -v\} \mid \|v\| = 1,\ v \in \mathbb{R}^3 \}$

The quotient maps shown are $v \mapsto \{v, -v\}$

The two sets "Pairs" have quotient topologies coming from ϕ, θ that might be different.
To show they are the same we show f and g are continuous.

The previous theorem says: g is continuous $\Leftrightarrow$ $g\theta$ is continuous. Here $g\theta = \phi i$ is continuous b/c ϕ and i are continuous.
$\therefore$ g is continuous.
Also f is continuous $\Leftrightarrow$ $f\phi$ is continuous.
If $V \subseteq$ Pairs is open in the topology from N ($\theta^{-1}(V)$ is open) I claim that

$(f\phi)^{-1}(V) = \theta^{-1}(V) \cup -\theta^{-1}(V)$. This is open so $f\phi$ and hence f are continuous. see picture
Thus the two constructions of RP^2 are homeomorphic.

? Worksheet 5 question 4 ?

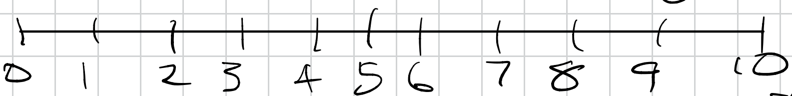
Wkst 5 qn 4. Show $\mathbb{RP}^1 \cong S^1$.

1. Do we want to use some mappings? (what should they be)

1.5 How do we define $\mathbb{RP}^1$ and S^1?

2. Do we want to use Theorem 5.2?

3. Where is all this on the difficulty scale?

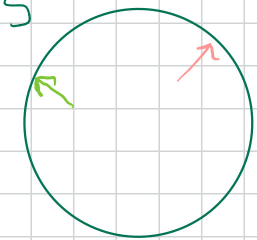


Easy some many Impossible.

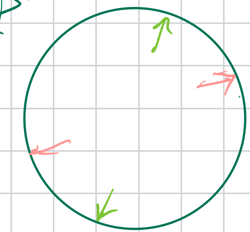
4. Have I just now done the problem?

Yes / No.

S^1



$\mathbb{RP}^1$



In $\mathbb{C}$ $z^2 \xleftarrow{f} z$

$z \xmapsto{g}$ choice of $\sqrt{z}$?

Better: $\{\pm\sqrt{z}\}$

Because f sends $\{\pm z\} \to z^2$

f induces a mapping $\hat{f}: \mathbb{RP}^1 \to S^1$.

$\hat{f}$ is continuous b/c the composite

$f: S^1 \xrightarrow[\text{quotient}]{} \mathbb{RP}^1 \xrightarrow{\hat{f}} S^1$ is continuous,

using Theorem 5.2.

Presumably something similar in the opposite direction also works.

Worksheet 4, 7b
Show that a subset U of a topological space X is open <=> U contains no point of its boundary.

Worksheet 4, 6a Show that a $B = \{\{a\},\{a,b\},\{c\}\}$ is a basis for a topology on $X = \{a,b,c\}$. Show that the open sets are:

Solution: do this by first verifying B is a basis

(i) $\bigcup_{V \in B} V = X$

(ii) $\forall\, x \in B_1 \cap B_2$ with $B_1, B_2 \in B$

$\exists\, B_3 \in B$ with $x \in B_3 \subseteq B_1 \cap B_2$

Check (ii). The only non-empty $\cap$ is

$\{a\} \cap \{a,b\}$ (apart from $\{a\} \cap \{a\}$ etc)
$= \{a\}$. Take $B_3 = \{a\}$ so
(ii) holds.

It follows that the unions of sets in B are topology.