Pre-class Warm-up!!!

Let R have the usual topology and let
x ~ y <=> x-y is rational.
Does the space R / ~ have

a. finitely many points
b. countably many points
c. uncountably many points ✓

The equivalence classes have the form $x+\mathbb{Q} := \{x+q \mid q \in \mathbb{Q}\}$ and are countable.
$\mathbb{R} = \bigcup(\text{equivalence classes})$ is uncountable.
There uncountably many equivalence classes.

Identify the quotient topology on R/~. Is it

a. The finite complement topology 0
b. The discrete topology 1
✓ c. The indiscrete topology 3
d. A metrizable topology 5
e. None of the above 0

Consider an open set $V \subseteq \mathbb{R}/\sim$
Let $f : \mathbb{R} \to \mathbb{R}/\sim$ be the quotient map
The preimage $f^{-1}(V) \subseteq \mathbb{R}$ is open.
If $V \neq \emptyset$ and $x \in f^{-1}(V)$ then $x+\mathbb{Q} \subseteq f^{-1}(V)$
Every point y in $x+\mathbb{Q}$ has some ball
$B_\epsilon(y) \subseteq f^{-1}(V)$, so $f^{-1}V \supseteq \underbrace{B_\epsilon(y)+\mathbb{Q}}_{=\mathbb{R}}$
We deduce $f^{-1}(V) = \mathbb{R}$. $V = \mathbb{R}/\sim$.

Chapter 6: Product spaces

Definition. Let X, Y be topological spaces.

The product topology on $X \times Y$ is the topology generated by the basis consisting of sets $U \times V$, where $U \subseteq X$ and $V \subseteq Y$ are open.

Proposition. The collection of sets U x V where U is open in X and V is open in Y is a basis for a topology.

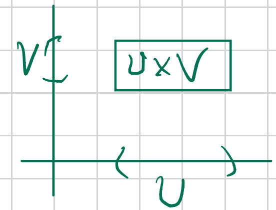
Check (i) $X \times Y$ is in the union of sets $U \times V$ (take $X = U$, $Y = V$).

(ii) If $U_1 \times V_1$, $U_2 \times V_2$ are products of open sets, $(x,y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$

then $(x,y) \in (U_1 \cap U_2) \times (V_1 \cap V_2)$

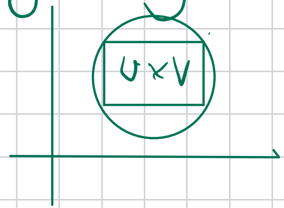
$\subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$.

Example: R^2 $= \mathbb{R} \times \mathbb{R}$ and if U, V are open sets in $\mathbb{R}$, then $U \times V$ is an open subset of $\mathbb{R}^2$ (usual topology)



Not all open sets in X x Y have the form U x V

We can get arbitrary open subsets of $\mathbb{R}^2$ by taking unions of $U \times V$ e.g.



unions fill up eg a disk.

Proposition. The product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the usual topology on $\mathbb{R}^2$.

Proof. Show that the basic sets in the product topology are open in the usual topology and the basic sets $B_\varepsilon(x,y)$ are open in the product topology.

Have we done the following?:

Proposition.

(a) The typical open set in X x Y has the form

$\bigcup_{i \in I} U_i \times V_i$ where $U_i \subseteq X$ $V_i \subseteq Y$ are open.

Yes / No

(b) This is a topology.

Yes / No

(c) (Theorem 6.3)

A subset W of X x Y is open <=> for all w in W there exist sets U , V with U open in X, V open in Y, and w in U x V which is contained in W.

Yes / No

Example. Let X = {a,b} with open sets
Ø, {a}, {a,b}.
How many open sets does X x X have?

a. 3
b. 4
c. 5
√d. 6
e. 7
f. 8
g. 9
h. 10
i. >20

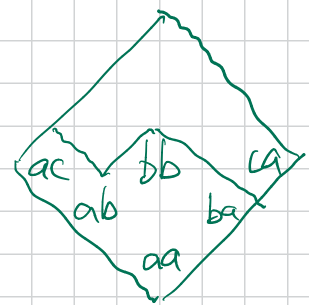
bb

ab ba

aa

products

Example. Let X = {a,b,c} with open sets
Ø, {a}, {a,b}, {a,b,c}.
How many open sets does X x X have?



open subsets = those closed by going down

← topology on the product poset

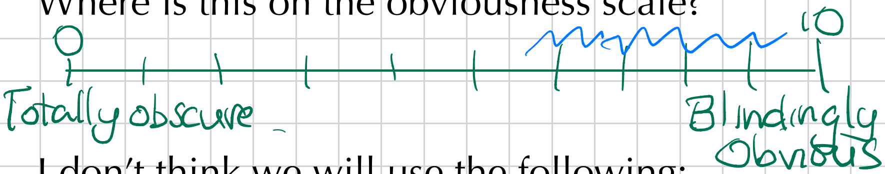
$$\begin{pmatrix} c \\ b \\ a \end{pmatrix} \times \begin{pmatrix} c \\ b \\ a \end{pmatrix}$$

Some properties of the product topology

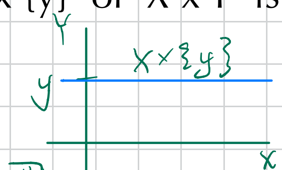
Let π_X : X x Y -> X and π_Y : X x Y -> Y be the projection maps: $\pi_X(x,y) = x$

$\pi_Y(x,y) = y$. $\pi_X^{-1}(U) = U \times Y$ which is open if U is open

Is it obvious that π_X and π_Y are continuous?
Where is this on the obviousness scale?



I don't think we will use the following:
Theorem 6.4
For all y in Y the subspace X x {y} of X x Y is homeomorphic to X.



Sketch: see the book.

The map $X\times\{y\} \hookrightarrow X\times Y \xrightarrow{\pi_X} X$ is continuous and bijective. It is open. See the book.

The universal mapping property.

Suppose A, X, Y are topological spaces.

$$\begin{tikzcd} & & X \\ A \arrow[urr, "f"] \arrow[drr, "g"'] \arrow[r, "\exists \text{ unique } h"'] & X\times Y \arrow[ur, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\ & & Y \end{tikzcd}$$

Theorem 6.5. Let f : A -> X and g : A -> Y be mappings of sets.
(a) There is a unique mapping h : A -> X x Y so that π_X h = f and π_Y h = g.
(b) f and g are continuous <=> h is continuous.