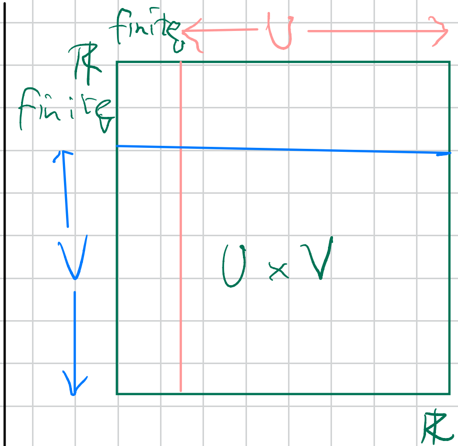
Pre-class Warm-up!!!

Can you remember the definition of the product topology?

Let the real numbers R have the finite complement topology.
Is the product topology on R x R also the finite complement topology?

Yes / No ✓



$$\mathbb{R}\times\mathbb{R} - U\times V = \text{finite}_U \times \mathbb{R} \cup \mathbb{R}\times\text{finite}_V$$

is infinite unless $U = V = \mathbb{R}$

$U \times V$ is open but need not have finite complement.

The universal mapping property.

Theorem 6.5. Let f : A -> X and g : A -> Y be mappings of sets, where A, X, Y are topological spaces.
(a) There is a unique mapping h : A -> X x Y so that π_X h = f and π_Y h = g.
(b) f and g are continuous <=> h is continuous.

Setup: We have projection maps π_X π_Y
$\pi_X(x,y) = x$, $\pi_Y(x,y) = y$

$$\begin{tikzcd} & A \arrow[r, "h"] \arrow[dl, "f"'] \arrow[drr, "g"] & X \times Y \arrow[dll, "\pi_X"'] \arrow[d, "\pi_Y"] \\ X & & Y \end{tikzcd}$$

Proof. (a) Necessarily $h(a) = (f(a), g(a))$ to get $\pi_X h = f$ $\pi_Y h = g$. This choice is unique.

(b) "$\Rightarrow$" Suppose f, g are continuous. Let $U \subseteq X$, $V \subseteq Y$ be open sets. We show $h^{-1}(U \times V)$ is open in A.

Now: $h^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V)$ is open. ($\cap$ 2 open sets). Obvious?
Because h^{-1}(union of open) = $\bigcup h^{-1}$(open)
This shows that h is continuous.

"$\Leftarrow$" Suppose h is continuous. Then $f = \pi_X h$ is continuous (composite of continuous mappings) as is g. $\square$

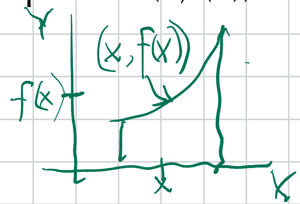
Why do we want to know this? Uniqueness of a product object is determined by the universal mapping property.

Application: worksheet 6 question 5

5,

The graph of $f : X \to Y$ is the set of points $(x,f(x))$ in $X \times Y$.

Let $F : X \to X \times Y$ be $F(x) = (x,f(x))$



a. If f is continuous then F is a homeomorphism between X and the graph of f.

b. If F is a homeomorphism between X and the graph of f then f is continuous.

Let $A = \{(x,f(x)) \mid x \in X\}$ be the graph.

We have

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[dr, bend right, "id_X"'] \arrow[drr, "f"] & X \times Y \arrow[d, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ & X & Y \end{tikzcd}$$

F is continuous $\Leftrightarrow$ id_X and f are continuous according to the last result.

This a connection between the last result and HW.

Given two objects X and Y, their product is an object $X \times Y$ together with homomorphisms π_X, π_Y so that whenever we have a diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "f"'] \arrow[drr, "g"] & X \times Y \arrow[d, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ & X & Y \end{tikzcd}$$

there exists a unique $h : A \to X \times Y$ so that $f = \pi_X h$ and $g = \pi_Y h$.

$$\begin{tikzcd} W & X \times Y \text{ or } X \sqcup Y \checkmark \arrow[l, "\exists ?"'] & \\ & X \arrow[ul] \arrow[u, "\iota_X"] & Y \arrow[ull] \arrow[ul, "\iota_Y"'] \end{tikzcd}$$

This is the coproduct of X and Y.

Activity: Worksheet 6 question 7.

Chapter 7: Compact spaces

The idea: theorems that work on some spaces and not on others.

Theorem. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Then f is bounded and achieves its bounds. (f has a maximum and f has a minimum)

Same result holds if $f : S^n \to \mathbb{R}^m$

It does not hold if $f : (0,1) \to \mathbb{R}$

e.g. $f(x) = \frac{1}{x}$ has no maximum on $(0,1)$

Theorem. If infinite sequence in $[0,1]$ has a convergent subsequence with limit in $[0,1]$.

The same holds for sequences in S^n

Example $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ has no convergent subsequence on $(0,1)$.

Definitions: cover, open cover, finite cover, sub cover

If $S \subseteq X$ is a subset of a topological space X a cover of S is a set of subsets V_i, $i \in I$ so that $S \subseteq \bigcup_{i \in I} V_i$.

The cover is an open cover if all V_i are open.

It is finite if I is finite (there are finitely many V_i).

A subcover of $\{V_i \mid i \in I\}$ is collection $\{V_j \mid j \in J\}$, $J \subseteq I$ for some subset J of I, that is a cover.

Definition: A subset S of a topological space X is compact $\iff$ every open cover of S has a finite subcover.

Definition: a subset S of a topological space X is compact <=> every open cover of S has a finite sub cover.

Examples (easy ones!): Take $X=S$ always.

X=R, { (n,n+3) | n in Z} - not compact.

X=R { [n,n+3] | n in Z}

X= (0,1) { (1/n, 1 - 1/n) | n in Z, n>0 } not compact

X= [0,1] { (1/n, 1 - 1/n) | n in Z, n>0 }

~~X~~= [0,1] $= S$ $X = \mathbb{R}$

{ (-1/4,1/4), (3/4,5/4), (1/n, 1 - 1/n) | n in Z, n>0 }

Is this an open cover? Yes / No	Does it have a finite sub cover? Yes / No
Yes	No
No	No
Yes	No
No	
Yes	Yes.

A technical result

Theorem 7.6 A subset S of X is compact <=> it is compact as a space given the induced topology.

Proof. "$\Rightarrow$" Suppose S is compact in the given definition.

Let $S = \bigcup_{i \in I} V_i$ where the $V_i \subseteq S$ are open in the induced topology.

This means each $V_i = W_i \cap S$ where W_i is open in X.

Now $S \subseteq \bigcup_{i \in I} W_i$ is compact so

$S \subseteq W_{i_1} \cup \cdots \cup W_{i_n}$, a finite union.

Now $S = (S \cap W_{i_1}) \cup \cdots \cup (S \cap W_{i_n})$

$= V_{i_1} \cup \cdots \cup V_{i_n}$

so S is compact in the induced topology.

"$\Leftarrow$" Suppose S is compact in the induced topology.

Let $S \subseteq \bigcup_{i \in I} W_i$, W_i open in X.

Then $S \subseteq S \cap \bigcup_{i \in I} W_i = \bigcup_{i \in I} (S \cap W_i)$

There is a finite subcover

$S = (S \cap W_{i_1}) \cup \cdots \cup (S \cap W_{i_n})$

Now $S \subseteq W_{i_1} \cup \cdots \cup W_{i_n}$

so S is compact. $\square$