Pre-class Warm-up!!!

Consider the following:
Every subset of R that is bounded above has a least upper bound.

Do you think this is true? Yes / No

Do you think you have seen it before? Yes / No

Do you think this is something you can prove?
Yes / No

What if we replace R be the rational numbers Q?
Does every subset of Q bounded above have a least upper bound? in $\mathbb{Q}$
Yes / No

R has an open cover consisting of { (-oo,5), (2,oo), (n,n+3) | n in Z}. This infinite open cover has a finite sub cover.
Does this show that R is compact?

Yes / No

The open cover $\{(n,n+3) \mid n \in \mathbb{Z}\}$ of $\mathbb{R}$ has no finite subcover so $\mathbb{R}$ is not compact.

A rather important result that motivates the theory, but will be subsumed as part of the Heine-Borel theorem.

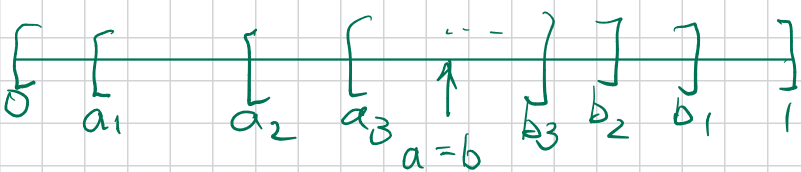
Theorem 7.7 The unit interval $[0,1]$ in $\mathbb{R}$ is compact.

Proof. We argue by contradiction. Let $[0,1] \subseteq \bigcup_{j \in J} U_j$ be an open cover and suppose it has no finite subcover.

Idea: We construct nested closed intervals

$$0 \le a_1 \le a_2 \le a_3 \le \cdots \le b_3 \le b_2 \le b_1 \le 1$$

with $b_i - a_i = \frac{1}{2^i}$



Divide $[0,1]$ in two: $[0,1] = [0,\frac{1}{2}] \cup [\frac{1}{2},1]$

One of $[0,\frac{1}{2}]$, $[\frac{1}{2},1]$ has no finite subcover by the U_j. Let $[a_1,b_1]$ be that interval.

Divide $[a_1,b_1]$ in two. One subinterval has no finite subcovers. Let $[a_2,b_2]$ be this interval.

Let $a=b$ be the least upper bound of the a_i = greatest lower bound of b_i.

$a=b$ b/c $b-a \le \frac{1}{2^i}$ $\forall i$.

$a \in U_j$ for some j. There exists $\epsilon > 0$ so that $B_\epsilon(a) \subseteq U_j$ b/c it's open.

Choose N so that $\frac{1}{2^N} < \epsilon$.

Then $[a_N,b_N] \subseteq B_\epsilon(a) \subseteq U_j$

so $\{U_j\}$ covers $[a_N,b_N]$.

Contradiction to $[a_N,b_N]$ has no finite subcover. $\square$

Some take-away results

Theorem
a. (7.8) A continuous image of a compact set is compact.
b. (7.10) A closed subset of a compact set is compact.
c. (7.11) The product of two compact sets is compact.

Theorem 7.8 Let $f: X \to Y$ be a continuous mapping between topological spaces. If X is compact, so is $f(X)$.

Proof. Let $f(X) \subseteq \bigcup_{j\in J} V_j$ be an open cover of $f(X)$.
Then $X = \bigcup_{j\in J} f^{-1}(V_j)$ is an open cover of X, so $X = f^{-1}(V_{j_1}) \cup \cdots \cup f^{-1}(V_{j_n})$ b/c X is compact.
Now $f(X) \subseteq V_{j_1} \cup \cdots \cup V_{j_n}$ is a finite subcover. $\square$

Applications:
Corollary 7.9 Every identification space $X/\sim$ is compact if X is compact.

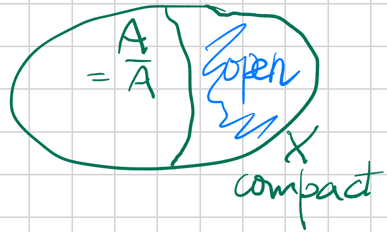
$[a,b]$ is compact.

If $X \cong Y$ and X is compact, so is Y.

Proof. The surjective map $f: X \to X/\sim$ is continuous. $\square$.

Example. $S^1 = [0,1]/\sim$ is compact.

Theorem 7.10 A closed subset of a compact set is compact.



Proof. Let $A \subseteq \bigcup_{j \in J} U_j$ be an open cover of A.
Now $(X-A) \cup \bigcup_{j \in J} U_j$ is an open cover of X, which is compact.
We can find $U_{j_1}, \dots, U_{j_n}$ so that $(X-A) \cup U_{j_1} \cup \cdots \cup U_{j_n}$ is a cover of X. Hence $A \subseteq U_{j_1} \cup \cdots \cup U_{j_n}$ is a finite subcover of A. $\square$

Theorem 7.11 Let X and Y be topological spaces. Then X and Y are compact <=> X x Y is compact.

Theorem 7.11 Let X and Y be topological spa

One direction is easier than the other. Which do you think it is?

"$\Rightarrow$"

"$\Leftarrow$" The projection $\pi_X : X \times Y \to X$ is continuous and surjective. A continuous image of a compact set is compact. Therefore if $X \times Y$ is compact so is X.

Which bit of the proof was hardest to understand?

Theorem 7.11 Let X and Y be topological spaces. Then X and Y are compact <=> X x Y is compact.

Proof "⇒" Suppose X, Y are compact, let $X \times Y = \bigcup_{j \in J} W_j$ be an open cover.

Write each $W_j = \bigcup_{k \in K} U_{jk} \times V_{jk}$ where U_{jk} is open in X, V_{jk} is open in Y. $\therefore X \times Y = \bigcup_{j,k} U_{jk} \times V_{jk}$

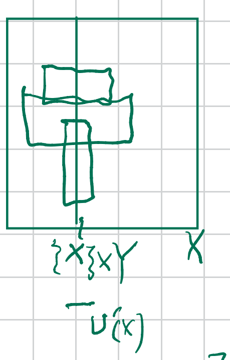
We find finitely many $U_{jk} \times V_{jk}$ that cover $X \times Y$, each contained in a W_j. These W_j cover $X \times Y$.

For each $x \in X$, $\{x\} \times Y \cong Y$, so is compact, so is covered by finitely many $U_{jk} \times V_{jk}$, so

$\{x\} \times Y \subseteq \bigcup \{ U_r(x) \times V_r(x) \mid r = 1, \ldots, n(x) \}$

Put $U'(x) = \bigcap_r U_r(x)$ so that

$\{ U'(x) \times V_r(x) \mid r = 1, \ldots, n(x) \}$ covers $\{x\} \times Y$



The $U'(x)$ covers X so finitely many suffice,

$X = \bigcup \{ U'(x_i) \mid i = 1, \ldots, m \}$

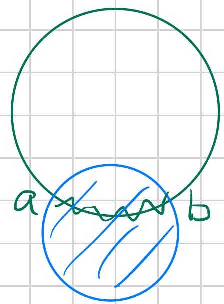
Now $\{ U'(x_i) \times V_{r_i}(x_i) \mid i = 1, \ldots, m, \ r_i = 1, \ldots, n(x_i) \}$ covers $X \times Y$. $\square$

Theorem (Heine-Borel) A closed and bounded subset of R^n is compact.

Proof. If $S \subseteq \mathbb{R}^n$ is closed and bounded $S \subseteq [-N, N]^n$ for some N, which is compact. A closed subset of a compact set is compact. S is compact $\square$

To show: the topology on $S^1 = \{v \in \mathbb{R}^2 \mid \|v\| = 1\}$ as an identification space $\mathbb{R}/\sim$ is the same as the subspace topology induced from $\mathbb{R}^2$.

For points $a, b \in S^1$ let $arc(a,b)$ be the subset of S^1 obtained by starting at a and going clockwise to b, omitting a and b.



1. These arcs are the intersections of disks in $\mathbb{R}^2$ with S^1, so are open subsets of S^1.

2. They form a basis for the induced topology

3. They are open in the quotient topology

4. Every open set in the quotient topology can be expressed as a union of these.

They are also a basis in the quotient topology

5. The two topologies are the same.