Which bit of the proof was hardest to understand?

Theorem 7.11 Let X and Y be topological spaces. Then X and Y are compact <=> X x Y is compact.

Proof "$\Rightarrow$" Suppose X, Y are compact, let $X \times Y = \bigcup_{j \in J} W_j$ be an open cover.

Write each $W_j = \bigcup_{k \in K} U_{jk} \times V_{jk}$ where U_{jk} is open in X, V_{jk} is open in Y. $\therefore$ $X \times Y = \bigcup_{j,k} U_{jk} \times V_{jk}$

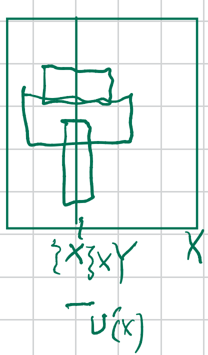
We find finitely many $U_{jk} \times V_{jk}$ that cover $X \times Y$, each contained in a W_j. These W_j cover $X \times Y$.

For each $x \in X$, $\{x\} \times Y \cong Y$, so is compact, so is covered by finitely many $U_{jk} \times V_{jk}$, so

$$\{x\} \times Y \subseteq \bigcup \{ U_r(x) \times V_r(x) \mid r = 1, \dots, n(x) \}$$

Put $U'(x) = \bigcap_r U_r(x)$ so that

$\{U'(x) \times V_r(x) \mid r = 1, \dots, n(x)\}$ covers $\{x\} \times Y$



The $U'(x)$ covers X so finitely many suffice,

$$X = \bigcup \{ U'(x_i) \mid i = 1, \dots, m \}$$

Now $\{U'(x_i) \times V_{r_i}(x_i) \mid i = 1, \dots, m,\ r_i = 1, \dots, n(x_i)\}$ covers $X \times Y$. $\square$

Theorem (Heine-Borel) A closed and bounded subset of R^n is compact.

Proof. If $S \subseteq \mathbb{R}^n$ is closed and bounded $S \subseteq [-N, N]^n$ for some N, which is compact. A closed subset of a compact set is compact. S is compact $\square$

Pre-class Warm-up!!!

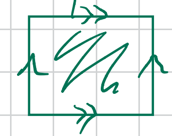
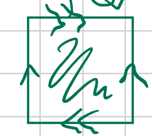
How many spaces do we know are compact at this point?

	Yes	No
a. Spaces with the discrete topology?		✓
b. Finite spaces	✓	
c. The circle S^1	✓	
d. The n-sphere S^n	✓	
e. Projective space RP^n	✓	
f. The open interval (0,1)		✓
g. The rational points {x in Q | 0 < x < 1 }		✓
h. Other compact spaces?		

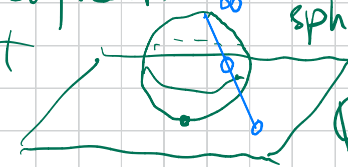
$\mathbb{R}P^n = S^n/\sim \qquad x \sim x \sim -x$.

is an identification space obtained from a compact space.

Other compact spaces

- X with the finite complement topology
- torus =  Klein bottle 
- one point compactification. ∞. Riemann sphere

The sphere is the 1-pt compactification of $\mathbb{C}$.



$\mathbb{C}$

continuous image of a compact set is compact

$[0,1]$ is compact

$[0,1]^n$ is compact. S^n is a closed subset of $[-1,1]^n$ or ...

not compact $(0,1) = \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 1-\frac{1}{n}\right)$.

5a.

(x_i) has a convergent subsequence $\iff$

$\exists\ y \in [0,1]$, $\forall \epsilon > 0$ $B_\epsilon(y)$ has ∞ly many points of x_i.

If not then $\forall\ x\ \exists\ \epsilon_x > 0$ so that $B_{\epsilon_x}(x)$ contains only finitely many points of the x_i. These $B_{\epsilon_x}(x)$ cover $[0,1]$. Finitely many do $[0,1] = \bigcup B_{\epsilon_x}(x)$. We deduce the sequence is finite.

Or: Infinitely many x_i lie in $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$. Call this interval $[a_1, b_1]$. Divide it in 2. Infinitely many x_i lie in one half interval. Call it $[a_2, b_2]$. We get $0 \le a_1 \le a_2 \le \dots \le b_2 \le b_1 \le 1$ and $a_i \le a = b \le b_i\ \forall i$. Now there is a convergent subsequence $\to a$.

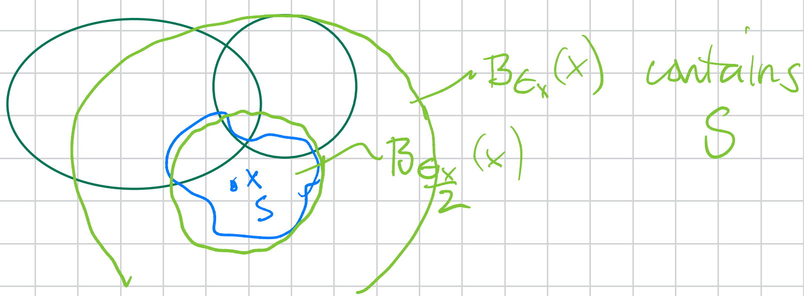
Worksheet 7 question 11.

Open cover $\{U_j \mid j \in J\}$. $\forall x \in X$ $\exists\ \epsilon_x$ so that $B_{\epsilon_x}(x) \subseteq U_j$ for some j.

These $B_{\epsilon_x}(x)$ cover X, so finitely many do. Given a set S of some small diameter, if $S \subseteq$ one of these fin many $B_{\epsilon_x}(x)$, it would be contained in a U_j.

Idea: take $\delta = \min\{\text{fin. many } \epsilon_x\}$.

How could we get $S \subseteq$ some $B_{\epsilon_x}(x)$?



Use instead $B_{\epsilon_x/2}(x)$. These cover X. Get a finite subcover $B_{\epsilon_{x_i}/2}(x_i)$ $\quad i = 1, \dots, n$.

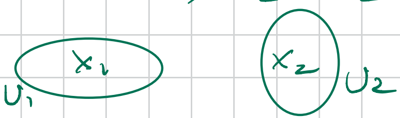
Let $\delta = \min\left\{ \frac{\epsilon_{x_i}}{2} \;\middle|\; i = 1, \dots, n \right\}$

If $y \in S$, $y \in B_{\epsilon_{x_i}/2}(x_i)$.

Now $S \subseteq B_{\epsilon_{x_i}}(x_i) \subseteq U_j$ for some j.

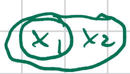
Chapter 8: Hausdorff spaces and separation axioms

Definition. A topological space X is Hausdorff
$\Leftrightarrow$ for each pair of points $x_1 \neq x_2$ in X
there exist open sets U_1, U_2 with
$x_1 \in U_1$, $x_2 \in U_2$, $U_1 \cap U_2 = \emptyset$.



Examples Metric spaces are Hausdorff
Put $d = d(x_1, x_2)$. Take
$U_1 = B_{d/3}(x_1)$ $U_2 = B_{d/3}(x_2)$.

More separation conditions:
T_0: For every pair of distinct points there is an open set containing one of them but not the other. If $x_1 \neq x_2$ there might be $x_1 \in U$ $x_2 \notin U$ or $x_2 \in U$ $x_1 \notin U$, be we don't which.

Example 

is T_0.

T_1: for every pair x,y of distinct points there are two open sets, one containing x but not y, the other containing y but not x.

T_2 = Hausdorff memorize this

T_3 X satisfies T_1 and for every closed subset F and every point x not in F there are two disjoint open sets, one containing F and the other containing x.

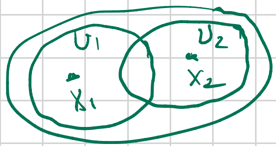
T_4: X satisfies T_1 and for every pair F_1,F_2 of disjoint ~~open~~ closed sets there are two disjoint open sets, one containing F F_1 and the other containing F_2.

Is there a difference between T_0 and T_1?
Which of these should we memorize? $T_2 \checkmark$ T_0, T_1?
Is it easy to see which of these axioms imply other axioms? $T_2 \Rightarrow T_1 \Rightarrow T_0$ is easy.
Can we construct spaces satisfying T_i but not T_{i+1}? With difficulty.
Can we construct spaces satisfying T_{i+1} but not T_i? No

Theorem 8.5 A space is T_1 <=> each point of X is closed.

Corollary 8.6 In a Hausdorff space, each point is a closed subset.

T_1



Proof of 8.5 "$\Leftarrow$" is easy. Suppose $\{x_1\}$, $\{x_2\}$ are closed. Take $U_1 = X - \{x_2\}$
$U_2 = X - \{x_1\}$.

"$\Rightarrow$". $\{x_1\} = \bigcap_{y \neq x \text{ in } X} X - U_y$

where $y \in U_y$ is an open set with $x_1 \notin U_y$.

This is an intersection of closed sets, so is closed. $\square$

Proof of 8.6: Hausdorff $\Rightarrow T_1$.

Theorem 8.7 A compact subset of a Hausdorff space is closed.