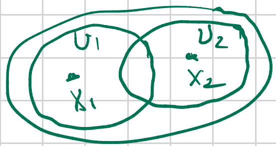
Theorem 8.5 A space is T_1 <=> each point of X is closed.

Corollary 8.6 In a Hausdorff space, each point is a closed subset.

T_1



Proof of 8.5 "$\Leftarrow$" is easy. Suppose $\{x_1\}$, $\{x_2\}$ are closed. Take $U_1 = X - \{x_2\}$
$U_2 = X - \{x_1\}$.

"$\Rightarrow$" $\{x_1\} = \bigcap_{y \neq x \text{ in } X} X - U_y$

where $y \in U_y$ is an open set with $x_1 \notin U_y$.

This is an intersection of closed sets, so is closed. $\square$

Proof of 8.6: Hausdorff $\Rightarrow T_1$.

Theorem 8.7 A compact subset of a Hausdorff space is closed.

Proof. Let $A \subseteq X$ where X is Hausdorff and A is compact.

We show $X - A$ is open.

Let $x \in X - A$. We show $\exists$ open set U with $x \in U$ and $U \subseteq X - A$

i.e $U \cap A = \emptyset$

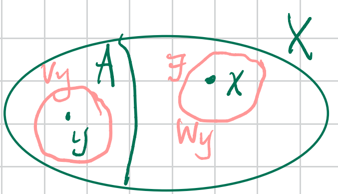
For each $y \in A$ we can find open V_y and W_y with $y \in V_y$ $x \in W_y$, $V_y \cap W_y = \emptyset$

The V_y, $y \in A$, is an open cover of A

so $A \subseteq V_{y_1} \cup \cdots \cup V_{y_n}$ (a finite subcover)

Now $x \in \bigcap_{i=1}^{n} W_{y_i} =: U$ is open and contains no point of A.

Thus $X - A$ is open and A is closed. $\square$



Pre-class Warm-up!!!

Can you remember what T_0, T_1, T_2 say?
What about T_{-1}?

Let X = R with the finite complement topology.
Does X satisfy any of these separation conditions?

	Yes	No
T_0	✓	
T_1	✓	
T_2		✓

Is the function f : R -> R^2 defined by
f(t) = (sin t, e^t) continuous?

Yes / No

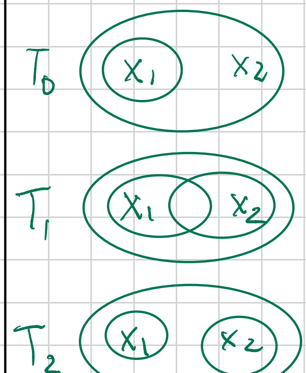
How do you know this?

Have you ever seen a proof of this in a calculus course?

Yes / No

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d] \arrow[dr] & X \times Y \arrow[dl, "\pi_X"'] \arrow[d, "\pi_Y"] \\ X & Y \end{tikzcd}$$

h is continuous
$\Leftrightarrow$ both $\pi_X h$ and $\pi_Y h$ are continuous.



≈ Hausdorff.

Theorem 8.8 A continuous bijective mapping from a compact space to a Hausdorff space is a homeomorphism.

Proof Let $f: X \to Y$ be continuous and bijective where X is compact, Y is Hausdorff.

We show f is open. Equivalently, we show f is closed, because f is bijective.

Let $V \subseteq X$ be a closed subset.
It is compact (closed subset of compact is compact)

Therefore $f(V)$ is compact. (cts image of compact is compact)

Therefore $f(V)$ is closed (compact subset of Hausdorff)
Thus f is a closed mapping. $\square$

Applications. We show that $\mathbb{RP}^2 = S^2/\sim$ where $v \sim \pm v$ is homeomorphic to $N/\sim = \{v \in \mathbb{R}^3 \mid \|v\| = 1, v_3 \geq 0\}/\sim$

where $v \sim v$ and $v \sim -v$ if $v_3 = 0$.

Construct.

$$\begin{tikzcd} S^2 \arrow[r, "f"] & S^2/\sim \\ N \arrow[u, hook, "\text{inclusion}"] \arrow[r, "g"] & N/\sim \arrow[u, "h"'] \end{tikzcd}$$

where f, g are quotient maps, h is induced by inclusion.

We see h is bijective. h is continuous $\Leftrightarrow$ hg is continuous $\Leftrightarrow$ $f \circ$ inclusion is cts which is so. Thus h is continuous.

We want to know h is open.
$N/\sim$ is compact (continuous image of N, which is compact)
$S^2/\sim$ is Hausdorff:

Geometrically given $\{v_1, -v_1\} \neq \{v_2, -v_2\}$ in $S^2/\sim$ we can find open sets $v_1 \in U_1$, $-v_1 \in -U_1$, $v_2 \in U_2$, $-v_2 \in -U_2$, so that the open sets are disjoint. $S^2/\sim$ is Hausdorff ✓ h is a homeomorphism

We can use this technique to show e.g.
$(0,1)^\infty \cong S^1$.

Define a mapping $f:(0,1)^\infty \longrightarrow S^1$ by

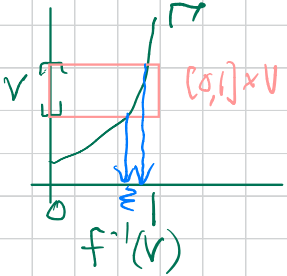
$$f(t) = \begin{cases} (\cos 2\pi t, \sin 2\pi t) & \text{if } t \in (0,1) \\ (1,0) & \text{if } t = \infty \end{cases}.$$

Show f is a continuous bijection.
$(0,1)^\infty$ is compact, S^1 is Hausdorff, so f is a homeomorphism.

Part of worksheet 7 question 10.
Let f : [0,1] -> R . Show that the graph of f is compact <=> f is continuous.
Give an example of a discontinuous function [0,1] -> R with a graph that is closed but not compact.

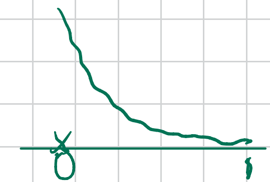
We do "$\Rightarrow$". Suppose the graph $\Gamma = \{(x,f(x)) \mid x \in [0,1]\}$ is compact. We show f is continuous by showing: if $V \subseteq \mathbb{R}$ is closed then $f^{-1}(V)$ is closed.
We use: $f^{-1}(V) = \pi_{[0,1]}\big(([0,1]\times V)\cap\Gamma\big)$

$[0,1]\times V$ is closed in $[0,1]\times\mathbb{R}$.
Thus $([0,1]\times V)\cap\Gamma$ is closed in Γ.
Thus $([0,1]\times V)\cap\Gamma$ is compact.
Thus $\pi_{[0,1]}(([0,1]\times V)\cap\Gamma)$ is compact b/c $\pi_{[0,1]}$ is continuous.
Thus it $= f^{-1}V$ is closed. $\square$



Example: $f(t) = \begin{cases} 0 & \text{if } t \le \frac{1}{2} \\ \frac{1}{t-\frac{1}{2}} & \text{if } t > \frac{1}{2} \end{cases}$

$g(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{1}{t} & \text{if } t > 0 \end{cases}$



Comments on worksheet 6 question 5: f : X ->
is continuous <=> F is a homeomorphism between X and the graph {(x,f(x) | x in X}.

Use technology! $F(x) = (x, f(x))$ is continuous b/c $x \mapsto x$ and $x \mapsto f(x)$ are continuous.

Theorem 8.9 A subspace of a Hausdorff space is Hausdorff.

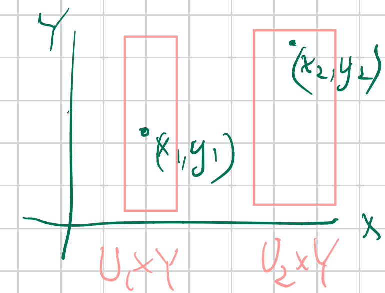
Theorem 8.10 Let X and Y be topological spaces.
Then X and Y are Hausdorff <=> X x Y is Hausdorff.

Question: Is either of these theorems obvious?
8.9 Yes ✓ / No
8.10 Yes / No

8.10 Take points $(x_1, y_1) \neq (x_2, y_2)$ in $X \times Y$.

If $x_1 \neq x_2$, take $x_1 \in U_1$, $x_2 \in U_2$ open sets with $U_1 \cap U_2 = \emptyset$.

Now $(x_1, y_1) \in U_1 \times Y$ $\quad U_1 \times Y \cap U_2 \times Y = \emptyset$
$(x_2, y_2) \in U_2 \times Y$



If $y_1 \neq y_2$ its similar. $\square$

Examples: $R/\sim$ where $x \sim y \iff x-y$ is rational.
X/A where A is not closed.

$R/\sim$ has the indiscrete topology.

The quotient map $\mathbb{R} \to R/\sim$ goes from a Hausdorff to a non-Hausdorff space.

Theorem 8.11 Let Y be a quotient space of X determined by a surjective mapping $f : X \to Y$. If X is compact Hausdorff and f is closed then Y is (compact) Hausdorff.

We won't do this.