Pre-class Warm-up!!!

True or false?

a. A closed subset of a closed set is closed.

✓True False

b. If S is a closed subset of R that is bounded above, then the least upper bound of S belongs to S.

✓True False

c. A closed subset of an open set is closed.

True False ✓

d. Let $\bar{S}$ = closure of S. Then $x \in \bar{S} \Leftrightarrow \forall$ open U with $x \in U$, $U \cap S \neq \emptyset$

✓True

Exam 2 on Wednesday: 50 mins in class

It is on Chapters 5 – 8 = quotient, compact, Product, Hausdorff spaces.

c. We can find open $S = (0,1) \subseteq \mathbb{R}$ with a closed subset such as $[0,1] \cap S = (0, \frac{1}{2}]$ that is not closed

$\mathbb{R}$

d. "$\Leftarrow$" Suppose the RHS. Argue by contradiction. Suppose $x \notin \bar{S}$. Thus $x \in X - \bar{S}$ which is open. Take $U = X - \bar{S}$. In fact $U \cap S = \emptyset$ so the condition on the RHS fails. Contradiction, so $x \in \bar{S}$.

"$\Rightarrow$" Suppose RHS fails, so $\exists$ open U, $x \in U$, $U \cap S = \emptyset$. Then $X - U \supseteq S$ and is closed so $X - U \supseteq \bar{S}$. Thus $x \notin \bar{S}$ so LHS fails.

Chapter 9: Connected spaces

A topological space X is connected if X has no subset that is both open and closed, apart from $\emptyset$ and X.

$[0,1] \cup [2,3]$ is not connected with the induced topology from $\mathbb{R}$.

$[0,1]$ is both open and closed, as is $[2,3]$ in this subspace.

A subset of X is connected if it is connected as a space with the induced topology.

Theorem 9.2. The following are equivalent for a space X.

1. X is connected.
2. X is not the union of two disjoint non-empty open subsets.

Proof. We show X is disconnected

$\iff$ $X = A \cup B$ where A, B are open, $A \cap B = \emptyset$, $A \neq \emptyset \neq B$

$\iff$ X has a subset $\emptyset \neq A$ that is open and closed, $X \neq A$

$\iff$ $X \supseteq A$ so that $A, X-A$ are open, $A \neq \emptyset$, $A \neq X$

$\iff$ $X = A \cup B$ where A, B are open, $A \neq \emptyset \neq B$.

Worksheet 9 questions 1, 2, 3.

1. The connected subsets of $\mathbb{Q}$ are $\emptyset$ and the 1-point sets $\{x\}$. Such sets are totally disconnected.

2. (i) (i) (ii) (ii)

totally obscure — blindingly obvious

3 Connected?

	Yes	No
1.	✓	
2.	✓	
3.		✓

Theorem 9.3. The interval $[0,1]$ in $\mathbb{R}$ is connected.

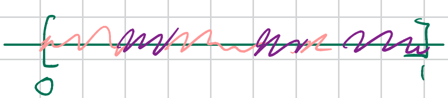
Proof. We argue by contradiction.
Let $[0,1] = A \sqcup B$ is a disjoint union of non-empty open sets. Suppose $0 \in A$.

Consider the least upper bound x of the set
$\{a \in A \mid a < b \ \forall b \in B\}$

Note that 0 lies in this set.
A is closed so $x \in A$.

Also, every $(x-\epsilon, x+\epsilon)$ intersects B in a non-empty set, because otherwise x would not be an upper bound.
Therefore $x \in \overline{B} = B$.

$A \cap B$ contains x so is non-empty, contradiction. □



Theorem 9.4. A continuous image of a connected space is connected.

i.e. Let $f: X \to Y$ be a continuous mapping. If X is connected so is $f(X)$.

Proof. We prove the contrapositive.
Suppose $f(X)$ is disconnected
$f(X) = A \cup B$ in the induced topology on $f(X)$, $A \cap B = \emptyset$, $A \neq \emptyset \neq B$ are open.
Then $X = f^{-1}(A) \cup f^{-1}(B)$ disconnects X. (disjoint union of open sets, neither $= \emptyset$)

□

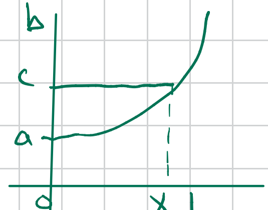
The continuous image of a ☺ is a ☺.
☺ can be compact, connected.

Corollary 9.5. If X and Y are homeomorphic then X is connected $\Leftrightarrow$ Y is connected.

Worksheet 9 question 5: the only connected subsets of R are intervals.

Application: the intermediate value theorem (10.1 in Kosniowski's book).

Theorem If $f : [0,1] \to \mathbb{R}$ is continuous $f(0) = a$, $f(1) = b$ then for all c with $a < c < b$ there exists $x \in [0,1]$ so that $f(x) = c$



Proof.

$f([0,1])$ is connected so is an interval. Thus since $a, b \in f([0,1])$ and $a < c < b$ then $c \in f([0,1])$.

$[0,1]$	Can remove 2 pts & it is connected; 3 pts must disconnect	$\mathbb{R}^2$	Remove n points and it is still connected.
S^1	Removing n points gives n pieces	Y	Can remove 1 pt, get 3 pieces

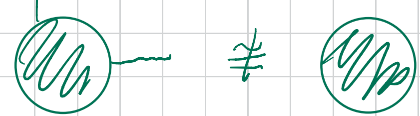
Application: showing that spaces are not homeomorphic. Worksheet 9 question 6.

Show that $(0,1)$, $(0,1]$, $[0,1]$, ◯, $\mathbb{R}^2$, Y are all non-homeomorphic.

The start of this. If we remove any single point from $(0,1)$ it is disconnected.

It is possible to remove 2 points from $[0,1]$ so that we get 2 connected components. We cannot get 1 piece like that. It is also possible to get 3 pieces. We can't get only one piece by removing 2 points.

Choose one of $[0,1]$, ◯, $\mathbb{R}^2$, Y and find a property that distinguishes it from the other 5.



is another example