Pre-class Warm-up!!!

Let $X = \{a,b,c\}$ with open sets
$\emptyset$, $\{a\}$, $\{a,b\}$, X

a. Is X connected? Yes No

b. Is X Hausdorff? Yes No

only open set cont. c

a b c

c. Is the mapping $f : X \to X$ given by
$f(a) = b$, $f(b) = c$, $f(c) = c$ continuous? Yes No

d. Is the mapping $g : X \to X$ given by
$f(a) = b$, $f(b) = a$, $f(c) = c$ continuous? Yes No

Which is the hardest question? c

$f^{-1}(\emptyset) = \emptyset$, $f^{-1}\{a\} = \emptyset$, $f^{-1}(\{a,b\}) = \{a\}$
$f^{-1}\{a,b,c\} = X$ are all open.

$f^{-1}\{a\} = \{b\}$ is not open.

Exam 2 is about

Quotient spaces

$$A \xrightarrow{f} A/\sim \xrightarrow{g} X$$

g is continuous $\iff gf$ is conts.

U is open $\iff f^{-1}(U)$ is open

f might or might not be closed, open

Weird examples like $\mathbb{R}/\sim$, $x \sim y \iff x - y \in \mathbb{Q}$.

Products

Basis for open sets: $U \times V$

$$A \xrightarrow{f} X \times Y \xrightarrow{\pi_X} X, \quad X \times Y \xrightarrow{\pi_Y} Y$$

π_X, π_Y are continuous

f is continuous $\Rightarrow f\pi_X$ and $f\pi_Y$ are continuous

e.g. $(\sin t, e^t)$ is continuous.

Compactness: closed subset of a compact set is compact.

Subsets of $\mathbb{R}^n$ are compact $\iff$ closed and bounded

$f(\text{compact})$ is compact

$(0,1) \subseteq \mathbb{R}^n$, $\mathbb{R}$ are not compact.

finite complement top is compact

Funny spaces: $\mathbb{Z}$, open sets are $\{n, n+1, \dots\}$ and $\mathbb{Z}$.

finite spaces are compact.

Hausdorff.

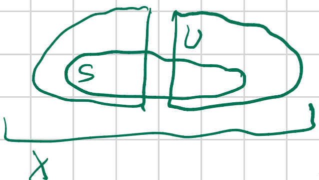
$= T_2 \Rightarrow T_1 \Rightarrow T_0$

$T_1 \iff$ one-point sets are closed.

e.g. if $A \xrightarrow{f} A/\sim$, if $f^{-1}(pt)$ is not closed then $A/\sim$ is not T_1, so not T_2.

compact subset of a Hausdorff space is closed

Lemma. If S is a connected subset of a topological space X and U is a subset that is both open and closed, then either S is contained in U or else S has empty intersection with U.



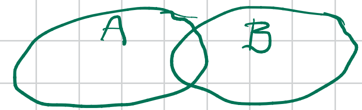
This S is not connected.

Proof. $S=(S\cap U)\cup(S\cap(X-U))$ disconnects unless $S\cap U=\emptyset$ or $S\cap(X-U)=\emptyset$. □

Theorem 9.6 If Y is the union of connected subspaces that all intersect non-trivially, then Y is connected.

Proof. Suppose $Y=\bigcup_{i\in I} Y_i$ and $\bigcap_{i\in I} Y_i\neq\emptyset$.
The Y_i are connected.
Suppose $U\subseteq Y$ is open and closed and $U\neq\emptyset$.
Some Y_i intersects U non-trivially so $Y_i\subseteq U$. $\bigcap_{i\in I} Y_i\subseteq U$.

Idea If connected sets intersect non-trivially their union is connected.



Every Y_i has non-empty intersection with U so is contained in U.
Therefore $Y=\bigcup_{i\in I} Y_i\subseteq U$
$Y=U$, and Y is connected. □

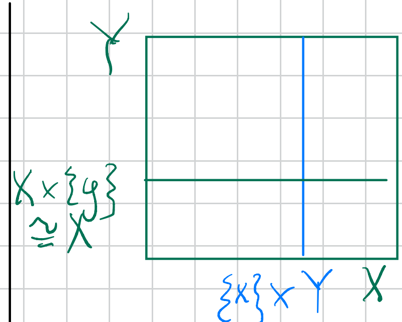
Theorem 9.7. Spaces X and Y are connected <=> X x Y is connected.

Proof. "$\Leftarrow$" The $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ are continuous and surjective. The continuous image of a connected set is connect(ed). Therefore $X = \pi_X(X \times Y)$ and Y are connected if $X \times Y$ is.

"$\Rightarrow$" Suppose X and Y are connected. For each $x \in X$ and $y \in Y$, $\{x\} \times Y$ and $X \times \{y\}$ are connected (homeomorphic to Y and X).

Now for any fixed $x \in X$, $\{x\} \times Y \cap X \times \{y\}$ is non empty, so $\{x\} \times Y \cup X \times \{y\}$ is connected.

$$X \times Y = \bigcup_{\substack{x \in X \\ y \in Y}} \{x\} \times Y \cup X \times \{y\}$$



which is a union of connected sets, which pairwise intersect non-trivially.

By a similar argument to 9.6 we see $X \times Y$ is connected:

If $U \subseteq X \times Y$ is an open and closed subset and any set $\{x\} \times Y \cup X \times \{y\}$ contains a point of U, it is contained in U, so every such set contains of point of U and is contained in U. Thus $U \supseteq X \times Y$ so $X \times Y$ is connected. $\square$