Pre-class Warm-up!!!

a. Have you heard of the following theorem before? Yes / No

Theorem. At any given moment there is a pair of antipodal points on the Earth with exactly the same temperature.

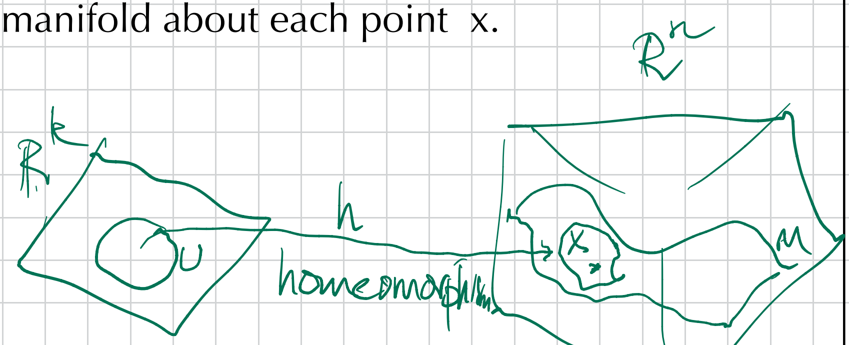
b. What kind of definition of a manifold have you seen before?

- Was the manifold always embedded in some space R^n ? Yes / No
- Was the manifold defined as the graph of a function? Yes / No
- Was the manifold described as the zero point set of some function? Yes / No
- Was it more complicated than this? Yes / No

Both Colley and Marsden-Tromba define parametrized manifolds.

Colley offers:
A k-manifold in R^n is a connected subset M of R^n such that
for all x in M there is an open set U in R^k
and a continuous one-to-one map X : U -> R^n
with x in X(U) contained in M.

That is, a k-manifold is locally a parametrized k-manifold about each point x.



h(U) is an open subset
of M (induced topology)

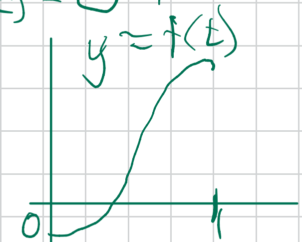
Which theorem was closest to the Intermediate Value Theorem?

a. Theorem A continuous image of a compact space is compact.

b. Theorem A continuous image of a connected space is connected.

Particular case of the IVT, with a different proof to the previous one:

Lemma 10.1 If $f: [0,1] \to \mathbb{R}$ is continuous, so that either $f(0) \geq 0$, $f(1) \leq 0$ or $f(0) \leq 0$ and $f(1) \geq 0$, then $\exists\, t \in [0,1]$ so that $f(t) = 0$.



Proof. Argue by contradiction. Assume $f(0) \neq 0 \neq f(1)$.

Suppose $f(t) \neq 0 \ \forall\, t \in [0,1]$.

Consider $g(t): [0,1] \to \{\pm 1\} = S^0$ defined by $g(t) = \dfrac{f(t)}{|f(t)|}$.

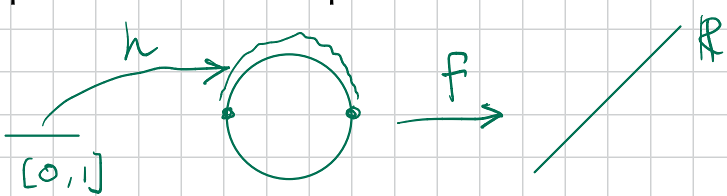
Then g is continuous and surjective (b/c $g(0)$ and $g(1)$ take opposite signs).

$[0,1]$ is connected but the image of g is not connected, contradiction. $\square$

Corollary 10.2. If f : [0,1] -> [0,1] is continuous then there is a point t in [0,1] such that f(t) = t.

Proof. Consider $h(t) = f(t) - t$.
Then $h(0) \geq 0$, $h(1) \leq 0$ so
$\exists\ t \in [0,1]$ with $h(t) = 0$, so that
$f(t) = t$. $\square$

Corollary 10.3. Every continuous mapping
S^1 -> R
sends at least one pair of diametrically opposite points to the same point.



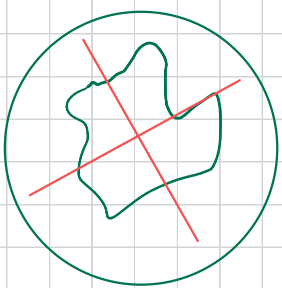
Proof. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.
and let $h : [0,1] \to S^1$ be
$h(t) = e^{i\pi t}$
Let $f : S^1 \to \mathbb{R}$ be a continuous mapping
Let $g(z) = f(z) - f(-z)$, $z \in S^1$
and consider $gh : [0,1] \to \mathbb{R}$.
Then $gh(0) = g(1) = f(1) - f(-1)$.
$gh(1) = g(-1) = f(-1) - f(1)$.
so $gh(1) = -gh(0)$. There is some
$t \in [0,1]$ so that $gh(t) = 0$, so
$f(h(t)) = f(-h(t))$. $\square$

Corollary 10.4. At any given moment, on any great circle, there is a pair of antipodal points on the Earth with exactly the same temperature.

Pancakes

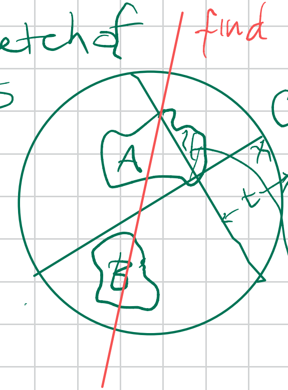
Theorem 10.5. Let A and B be bounded subsets of R^2. There is a line in R^2 that divides each region exactly in half by area.

Theorem 10.6. If A is a bounded region in the plane then there exists a pair of perpendicular lines that divide A into four parts, each of the same area.



Sketch of find this line.

10.5



Circle radius N.

Scale to have diameter 1.

$g_0(t)$ = area on one side of the perpendicular line

$g_1(t)$ = area on other side

$g_1(t) - g_0(t) : [0,1] \longrightarrow \mathbb{R}$

has a zero for some $t \in [0,1]$.

A is cut in half like this. For each x we get such t: $u(x)$

Do the same for pancake B giving $v(x)$

Consider $v - u : S^1 \to \mathbb{R}$.

Find x so that $v(x) - u(x) = 0$.

For that x the two values of t are the same and we cut both A and B in half at the same time. $\square$

Chapter 11. Manifolds and surfaces

Definition. An n-dimensional manifold is a Hausdorff space M such that for every $x \in M$ there is an open neighborhood U_x with $x \in U_x$ and $U_x \cong \mathbb{R}^n$. The open disk $(D^n)^o$ is homeom. to $\mathbb{R}^n$.

Example: 0-dimensional manifolds

$\mathbb{R}^0$ is a single point. If M is a 0-manifold has: $\forall\, x \in M$, $\{x\}$ is open. M has the discrete topology.

Example (not done like this in the book). Subsets of R^d that are locally the graphs of continuous functions R^n -> R^{d-n} are manifolds.

If $U \subseteq \mathbb{R}^n$ is open and $f: U \to \mathbb{R}^{d-n}$ is continuous then the graph of f

$$\{(x, f(x)) \mid x \in U\} = \Gamma \subseteq \mathbb{R}^n \times \mathbb{R}^{d-n} \cong \mathbb{R}^d$$

is homeomorphic to U.

In U, x has an open neighborhood $\cong \mathbb{R}^n$, so $(x, f(x))$ also has such a neighborhood in Γ.

Locally ... means that $\forall\, m \in M$ $\exists$ an open neighborhood $m \in V \subseteq M$ so that $V \cong$ the graph of a continuous function.

Example $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a 1-manifold.

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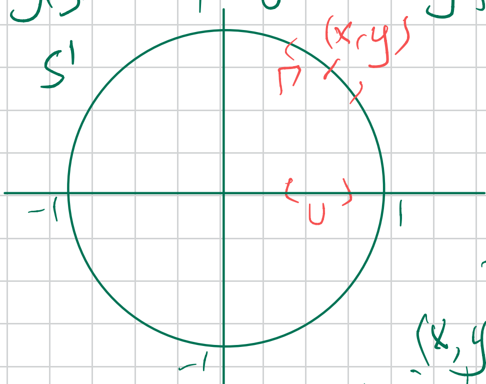
is a 1-manifold.
Consider the functions

$f^+(x) = +\sqrt{1-x^2}$ $\quad f^- = -\sqrt{1-x^2}$

$g^+(y) = +\sqrt{1-y^2}$ $\quad g^-(y) = -\sqrt{1-y}$



If $y>0$ and $(x,y) \in S^1$ then around (x,y) There is a neighborhood so that S^1 in that nhd is the graph of f^+ on some open subset of $\mathbb{R}^1$.

If $y<0$ we can use f^-

If $x>0$ we use g^+

If $x<0$ $\quad g^{-1}$

Higher dimensional spheres are manifolds, using

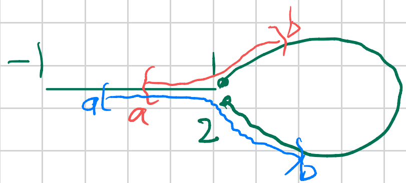
$f^+(x,y) = +\sqrt{1-(x^2+y^2)}$ $\quad f^{-1}$ etc.

gives $S^2 \subseteq \mathbb{R}^3$.

Example (taken from the book). Why do we need the Hausdorff axiom?

Consider the space $X = (-1, 2]$ with basis for a topology consisting of

- open intervals (a, b) $-1 \leq a < b \leq 2$
- $(a, 1) \cup (b, 2]$



1 ∉ the blue open set
2 ∉ the pink open set.

This space is not Hausdorff: all open sets containing 1 contain $(1-\epsilon, 1)$ for some ϵ, as do all open sets containing 2. We can't separate 1 and 2 by open sets.

Every point in the space has an open neighborhood homeo to $\mathbb{R}$ (or $(0, 1)$).

Questions: a. If you remove a single point from this space, does it necessarily disconnect it?
Yes / No   remove 1 or 2.

b. Construct a space that is locally homeomorphic to R with 3 points that cannot be separated from each other by open sets.