The relation T#P = P#P#P.

Rules for transformation of surface diagrams:

1. Each letter appears exactly twice, in the form y or y^{-1} = Y
2. Relabel the words (including a <-> a^{-1} = A).
3. Cyclically permute words: abc <-> bca.
4. Flip words: w = aBc <-> W = CbA.
5. (Un)cancel: yaAz <-> yz
6. (Un)split words: wCcx <-> wC and cx
7. (Un)write strings as single letters, such as: w ab x BA y <-> w c x C y
8. From a word w x where the letters in w are disjoint from those in x, we get a connected sum.

Useful move $ay_1ay_2 \to ay_1c^{-1}cay_2$

$\to cy_1^{-1}a^{-1}ay_2c \to cy_1^{-1}y_2c$

$\to ccy_1^{-1}y_2$.

Proof $P\#T = P\#P\#P$

$P\#T = ccaba^{-1}b^{-1} = ca\underbrace{bm^{-1}}_{\text{rotate}}m\underbrace{a^{-1}b^{-1}c}_{\text{invert}}$

$\to abm^{-1}c\,c^{-1}bam^{-1} = abm^{-1}bam^{-1}$

$= a\underbrace{bnb}_{y_1}a\underbrace{n}_{y_2}$

$\to \underbrace{cc}_{P}\,\#\,\underbrace{b^{-1}n^{-1}b^{-1}n}_{K}$

Worksheet 11 qn 3: $y_0ay_1ay_2 \sim y_0ab^{-1} \sqcup by_1ay_2$

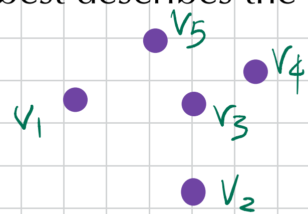
Invert y_0ab^{-1}, rotate both parts: $\sim ba^{-1}y_0^{-1} \sqcup by_1ay_2$

$\sim y_0^{-1}ba^{-1} \sqcup ay_2by_1 \sim y_0^{-1}by_2by_1$

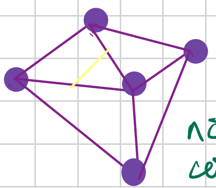
$\sim by_2by_1y_0^{-1}$

Pre-class Warm-up!!!!

Which picture best describes the convex hull of the 5 points:



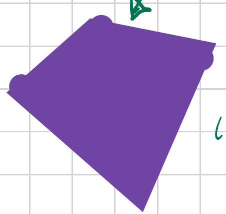
a.



not convex

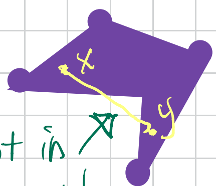
c.

convex hull



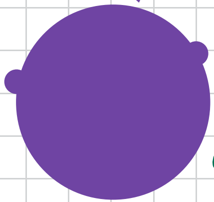
kite is convex

b.



not in the subset

d.



disk is convex

e. None of the above.

How obvious is the following?

Theorem. Every compact surface can be made by taking a polygonal piece of paper and glueing up the edges in the kind of way we have been doing.

totally obscure — blindingly obvious

How many words on this screen are new to you?

a. 0 b. 1 c. 2 d. ≥ 3

Definition. A subset of $\mathbb{R}^n$ is convex $\Leftrightarrow$ $\forall x, y$ in the subset, the straight line segment joining x and y lies in the subset.

The convex hull of a subset $S \subseteq \mathbb{R}^n$ is the smallest convex set containing those points. It consists of the points $\{ \sum \lambda_i v_i \mid 0 \leq \lambda_i \leq 1, \sum \lambda_i = 1 \}$

Constructing surfaces: simplicial complexes
Useful book:
Munkres: Elements of algebraic topology

Geometric simplicial complexes.

Definition. A geometric n-dimensional simplex in R^m is the convex hull of n+1 points in general position.

The vertices of the simplex are the n+1 points

What is convex? hull? general position?

n+1 points are in general position if they do not lie in an (n-1)-dimension affine subspace
$=$ a subset $w+V$ where $w \in \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ is an n-dimension vector subspace.

Examples: not general position; standard basis vectors.

In $\mathbb{R}^2$

not in general position

in general position

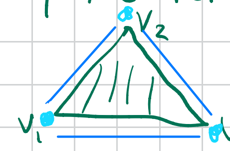
a 2-simplex.

Formula for the set of points in a simplex. A simplex is determined by its vertices.

The subsimplices, or faces, of a simplex.

These are the subsets that are the convex hulls of subsets of the vertices.

Example



The 2-simplex determined by $\{v_1, v_2, v_3\}$ has faces $\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_1, v_3\},$
$\{v_2, v_3\}, \{v_1, v_2, v_3\}\}$

where e.g. $\{a, b\}$ indicates the convex hull of those points.

Definition. A (geometric) simplicial complex is the union of a collection of simplices in R^n so that

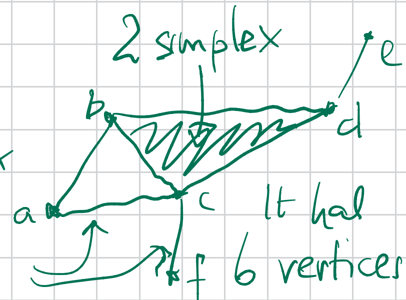
(1) for every simplex, all its faces are in the collection.

(2) Every pair of simplices intersects in a common face of those simplices or is empty.

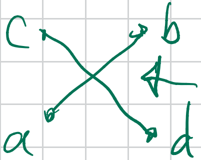
Its dimension is the largest n so that n-simplices appear.

Examples.

A simplicial complex

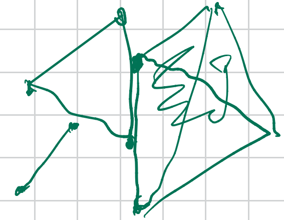


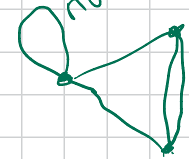
$\subsetneq R^n$



not allowed.

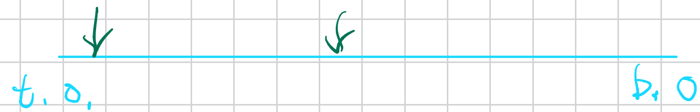
Not simplicial complexes

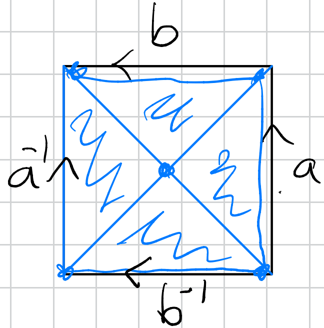


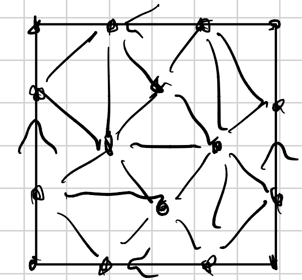


Is it obvious?

If we take the m vertices of a simplicial complex to be the standard basis vectors $(0,\ldots,0,1,0,\ldots)$ in R^m, then simplices only intersect in subsimplices?







probably works.

Is this a simplicial complex homeomorphic to the torus?

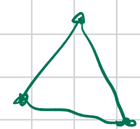
Yes / No

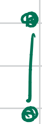
No: The 4 apparent vertices are a single vertex in the torus.

The 'simplicial complex' only has 2 vertices. This doesn't even allow 2-simplices.

Note: each edge becomes a loop in the torus

To get a simplicial complex homeomorphic to S^1 we need ≥ 3 vertices

 $\cong S^1$

 won't give S^1

Definition. An abstract simplicial complex is

- A set of things called vertices $\{v_1, \dots, v_n\}$
- A collection of subsets of $\{v_1, \dots, v_n\}$ with the property

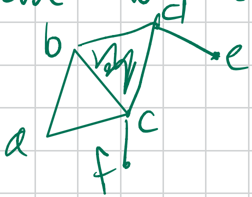
  If subset S is in the collection and $T \subseteq S$ then T is in the collection.

A subset S in this collection of size $n+1$ is called an n-simplex.

Vertices = 0-simplices.

Connection between abstract simplicial complexes and geometric simplicial complexes:

Given a geometric s.c. we get an abstract s.c.



$\rightsquigarrow \{a, b, c, d, e, ac, ab, bc, bd, cd, de, bcd\}$

Given an abstract s.c. we can produce a geometric s.c. in $\mathbb{R}^n$ where n = no. of vertices

vertices = standard basis vectors $e_1, \dots, e_n$

geometric simplices are the convex hulls of the S where S is an abstract simplex.