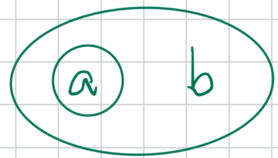
Pre-class Warm-up!!!

Let X be the topological space $X = \{a, b\}$ with open sets $\varnothing, \{a\}, X$.
Do you think X is path connected?

Yes / No



Consider

$f : [0,1] \to X$ where

$$f(t) = \begin{cases} a & \text{if } t < \frac{1}{2} \\ b & \text{if } t \geq \frac{1}{2} \end{cases}$$

Then f is continuous, so is a path from a to b.

Theorem 12.4. The continuous image of a path connected space is path connected.

Proof. Let $g: X \longrightarrow Y$ be continuous, where X is path connected. We show $g(X)$ is path connected.
Let $u, v \in g(X)$ and write
$u = g(a)$, $v = g(b)$ for some $a, b \in X$
We can find a path $f: [0,1] \to X$ with $f(0) = a$, $f(1) = b$.
Now $gf: [0,1] \to g(X)$ is a path from u to v. Thus $g(X)$ is path connected. $\square$

Corollary 12.5. If X and Y are homeomorphic and X is path connected so is Y.

Theorem 12.6. If X is the union of path connected spaces that overlap, then X is path connected.

Proof. Suppose $X = \bigcup_{i \in I} Y_i$ and $c \in \bigcap_{i \in I} Y_i$, where the Y_i are path connected. If $a, b \in X$ then $a \in Y_i$, $b \in Y_j$ for some i, j.
Find a path f in Y_i from a to c and a path g in Y_j from c to b.
Now $g * f$ is a path in X from a to b.
Therefore X is p-c. $\square$

Theorem 12.7. X and Y are path connected <=> so is X x Y.

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<=> so is X x Y.

Proof. Same as for connectivity.

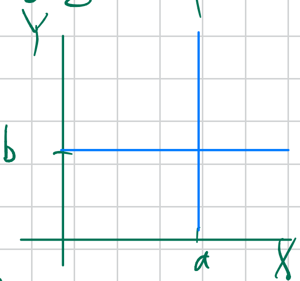
"$\Rightarrow$" Assume X and Y are pc.

Then for each $a \in X$, $b \in Y$ the spaces $\{a\} \times Y$ and $X \times \{b\}$ are p.c.

being homeomorphic to Y and X resp.

Therefore $\{a\} \times Y \cup X \times \{b\}$ is p.c by 12.6

$X \times Y$ is the union of these connected subspaces, and each pair of them has non-empty intersection.



Therefore $X \times Y$ is p.c.

"$\Leftarrow$" If $X \times Y$ is p.c. so are X, Y because they are continuous images of $X \times Y$. $\square$

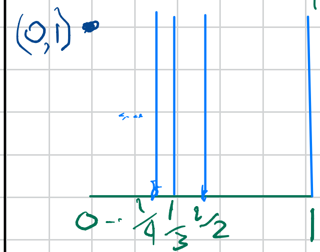
Theorem 12.8. Every path connected space is connected. Not every connected space is path connected.

Proof of the first sentence:
Suppose X is path connected but not connected, so $X = A \sqcup B$ is the disjoint union of non-empty open sets A and B.
Let $a \in A$ and $b \in B$. There is a path $f : [0,1] \longrightarrow X$ with

$f(0) = a$ and $f(1) = b$.

Now $[0,1] = f^{-1}(A) \sqcup f^{-1}(B)$ is a disjoint union of non-empty open sets, so $[0,1]$ is not connected, which is absurd. $\square$

The flea and comb is a connected space that is not path connected



is the union of the closed line segments shown, together with $(0,1)$. $\subseteq \mathbb{R}^2$

This is not path connected: there is no path joining the flea to the rest of the comb. Read the book.

The space is connected. Suppose it is $A \sqcup B$, a disjoint union of open sets.

The comb is contained in one of A and B because it's connected. The flea must be contained in the other.
This means we can an open set containing the flea, containing no point of the comb.
In fact every $B_\varepsilon(0,1)$ contains a point of the comb. Contradiction. It's connecte

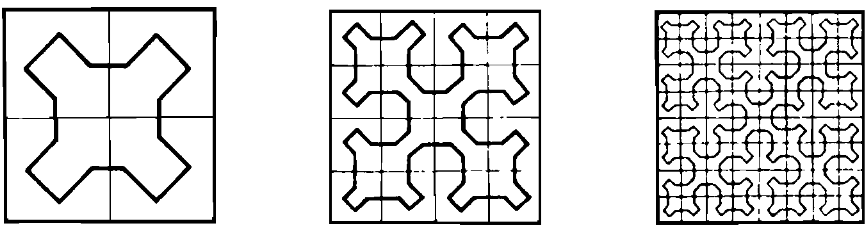
$C = \{ (x,y); 0 < x \leq 1, y = ½ \sin(\pi/x) \}$.

is path connected but not locally path connected.

(p) Let $Z = Y \cup D \in R^2$ where Y is as above in (o) and D is the circle $\{ (x-1)^2 + y^2 = 1 \}$. Prove that Z is path connected but not locally path connected.

We end this chapter with a strange path. This is a path $f: I \to I^2$ which is surjective. Such examples are referred to as 'space filling curves'. They were first invented by G. Peano in about 1890. The path f is defined as the limit of paths $f_n: I \to I^2$. The first three are illustrated in Figure 12.4. The reader should have no trouble in visualizing the n-th step. After n steps every point of the square I^2 lies within a distance of $(½)^n$ of a point in $f_n(I)$. In the limit we get a continuous surjective map $f: I \to I^2$. Note that at any finite stage the continuous map $f_n: I \to I^2$ fails to be injective only at $\{0\}$ and $\{1\}$ in I. In fact $\mathring{I}$ and $f_n(\mathring{I})$ are homeomorphic. This is certainly not true in the limit.

Figure 12.4



$f_1(I)$ $f_2(I)$ $f_3(I)$

Worksheet 10

4.* (a) Let M be an n-manifold and let B be a open subspace of M which is homeomorphic to an open ball of some radius. Because $B \cong \mathbb{R}^n \cong S^n - \{(0,0,\ldots,0,1)\}$ we have a homeomorphism $g : B \to S^n - \{(0,0,\ldots,0,1)\}$. Define $f : M \to S^n$ by

$$f(x) = \begin{cases} g(x) & \text{if } x \in B, \\ (0,0,\ldots,0,1) & \text{if } x \in M - B \end{cases}$$

Prove that f is continuous.

Some solutions to the questions on worksheet 10 are posted online.

We show that if $U \subseteq S^n$ is open then $f^{-1}(U)$ is open in M.

Case 1. $e_n = (0,\ldots,0,1) \notin U$.
Then $f^{-1}(U) = g^{-1}(U)$ which is an open subset of B. Because B is open in M, $g^{-1}(U)$ is open in M.

Case 2. $e_n \in U$. Consider $S^n - U$, which is a closed subset of the compact space S^n, so it is compact.

$f^{-1}(S^n - U) = g^{-1}(S^n - U)$ and this is also compact, because g is a homeomorphism. Now $f^{-1}(S^n - U)$ is closed in M, because M is Hausdorff.

It follows that $f^{-1}(U) = M - f^{-1}(S^n - U)$ is open in M.

Thus in both cases $f^{-1}(U)$ is open in M, so f is continuous.