Math 5345H **Worksheet 10** **11/4/2025**

Upload to Gradescope the starred questions 1, 4, 7 from Worksheet 10 before the end of the day on Wednesday 11/12/2025

1.* Let $f : [0,1] \to [0,1]$ be a continuous function. Show that there is a point $t \in [0,1]$ for which $f(t) = 2t$.

2. Show that an open subset of an n-manifold is also an n-manifold.

3. Show that an n-manifold may be equivalently be defined as a Hausdorff space M such that each point of M has a neighborhood homeomorphic to the closed n-disk D^n.

4.* (a) Let M be an n-manifold and let B be a subspace of M which is homeomorphic to an open ball of some radius. Because $B \cong \mathbb{R}^n \cong S^n - \{(0,0,\ldots,0,1)\}$ we have a homeomorphism $g : B \to S^n - \{(0,0,\ldots,0,1)\}$. Define $f : M \to S^n$ by

$$f(x) = \begin{cases} g(x) & \text{if } x \in B, \\ (0,0,\ldots,0,1) & \text{if } x \in M - B \end{cases}$$

Prove that f is continuous.

(b) Assume now that M is compact, show that there is a finite cover $\{B_1, B_2, \ldots, B_m\}$ of M by open balls B_i, so that we have mappings $f_i : M \to S^n$, one for each B_i as constructed in (a). Define $f : M \to (S^n)^m$ by $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ and compose f with the inclusion $(S^n)^m \subseteq (\mathbb{R}^{n+1})^m$. Show that this gives a homeomorphism from M to a subspace of $(\mathbb{R}^{n+1})^m$.

5. How obvious is the following for surfaces S_1, S_2 and S_3?
(i) $S_1 \# S_2 \cong S_2 \# S_1$.
(ii) $(S_1 \# S_2) \# S_3 \cong S_1 \# (S_2 \# S_3)$
(iii) $S^2 \# S_1 \cong S_1$.

6. A *simple closed curve* in M is a subset homeomorphic to S^1.
(a) Show that a torus T has two distinct (but not disjoint) simple closed curves C_1, C_2 such that $T - (C_1 \cup C_2)$ is connected.
(b) Show that a torus T does not have three distinct simple closed curves C_1, C_2 and C_3 such that $T - (C_1 \cup C_2 \cup C_3)$ is connected.
(c) Generalize (a) and (b) to a sphere with n handles.

7.* In taking the connected sum of a surface M and a torus T we remove a ball from M and a ball from T and glue the two together along their boundaries, choosing a direction of the boundary of M to align with the boundary of T. Suppose we change the direction of glueing of the boundary of T, but not the direction of the boundary of M, so that now the glueing is done with the opposite alignment to what happened before. Show that the new connected sum is homeomorphic to the old connected sum.