Math 5345H **Worksheet 12** **11/16/2025**

No homework is due on Wednesday 11/26/2025

1. Consider abstract simplicial complexes Δ_1 and Δ_2 with six vertices $\{a, b, c, d, e, f\}$. We will describe the simplices of each Δ_i as sets of vertices, such as $\{a, c, e, f\}$ (this is a 3-simplex). To economize on notation we will write this set simply as $acef$. As an example of the notation, $aefc$ denotes the same 3-simplex.

(a) Let Δ_1 have 2-simplices $\{abc, bcd, cde, def, efa, fab\}$ and 1-simplices the collection of all subsets of size 2 of these 2-simplices, this (together with the six vertices) being a complete list of the simplices of Δ_1. How many 1-simplices does Δ_1 have? Identify Δ_1 as a space we have seen before, giving its name.

(b) Δ_2 has 2-simplices $\{abc, bcd, cde, def, efb, fab\}$ and 1-simplices the collection of all subsets of size 2 of these 2-simplices, this (together with the six vertices) being a complete list of the simplices of Δ_2. How many 1-simplices does Δ_2 have? Identify Δ_2 as a space we have seen before, giving its name.

2. Find a specific simplicial complex that is homeomorphic to a 2-sphere. Describe it both as a piece of paper divided up as triangular regions, with a recipe for glueing the edges together; and also as an abstract simplicial complex. In the description as an abstract simplicial complex, first list all the vertices, then list all the subsets of the vertices that are the vertices of simplices in your simplicial complex. Can you find such a simplicial complex with seven 2-simplices or fewer?

3. Same question as 2 with the torus instead of the 2-sphere. This time, can you find such a simplicial complex with nine 2-simplices or fewer?

4. Same question as 2 with the projective plane instead of the 2-sphere. Can you find such a simplicial complex with nine 2-simplices or fewer?

5. Same question as 2 with the Klein bottle plane instead of the 2-sphere. Can you find such a simplicial complex with nine 2-simplices or fewer?

6. The *Euler characteristic* of a finite simplicial complex C may be defined to be

$$\chi(C) = n_0 - n_1 + n_2 - n_3 + \cdots = \sum_{n \geq 0} (-1)^i n_i$$

where n_i is the number of i-simplices in the simplicial complex. It is a fact that if two simplicial complexes are homeomorphic as topological spaces then their Euler characteristics are the same.

(a) Show that if a surface is made up of polygonal pieces of paper (instead of triangles), glued along their edges, then the Euler characteristic of the surface equals $m_0 - m_1 + m_2$ where m_0 is the number of vertices, m_1 is the number of edges, m_2 is the number of polygonal faces. For example, the 2-sphere can be realized as the surface of a cube, which has square faces, with $m_0 = 8$, $m_1 = 12$, $m_2 = 6$, so its Euler characteristic equals $8 - 12 + 6 = 2$.

(b) Calculate the Euler characteristics of a sphere, a projective plane, a Klein bottle and a torus.

(c) Show that if M_1 and M_2 are surfaces then $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2$.

(d) Calculate the Euler characteristics of the surfaces $T \# T \# \cdots \# T$ and $P \# P \# \cdots \# P$ where T is the torus and P the projective plane.

7. (a) Prove that any space with the indiscrete topology is path connected.

(b) Consider the topological space $X = \{a, b\}$ with open sets $\emptyset, X, \{a\}$. Is X path connected?

8. (This extends question 11 on Worksheet 9.) This space and the space in the next question are the kind of weird things that are often considered at this point in this kind of course. Let A and B be subsets of $\mathbb{R}^2$ defined by

$$A = \{(x, y) \mid x = 0,\ -1 \le y \le 1\}$$
$$B = \{(x, y) \mid 0 \le x \le 1,\ y = \cos(\pi/x)\}$$

Prove that $X = A \cup B$ is connected but not path connected. (Draw a picture. Prove that A and B are connected. If $X = U \cup V$ where U, V are open and closed in X, figure out what they must be. Finally assume that some point of A is in U.)

9. (This extends question 12 on Worksheet 9.) Let A and B be subsets of $\mathbb{R}^2$ defined by

$$A = \{(x, y) \mid 1/2 \le x \le 1,\ y = 0\}$$
$$B = \{(x, y) \mid 0 \le x \le 1,\ y = x/n \text{ where } n \in \mathbb{N}\}$$

Prove that $X = A \cup B$ is connected but not path connected.

10. Let $\sim$ be the relation on the points of a space X defined by saying that $x \sim y$ if and only if there is a path in X joining x and y.

(a) Prove that $\sim$ is an equivalence relation. The equivalence classes are the *path connected components* of X.

(b) Consider the quotient space $X/\sim$. Find how many points this space has and list all the open sets in the cases of questions 8, 9, or the example of the 'flea and comb'.