Math 5345H **Worksheet 2** **9/8/2025**

Upload to Gradescope the starred questions 3, 6, 9, 12 from Worksheet 2 before the end of the day on Wednesday 9/17/2025

1. Consider a function $d : A \times A \to \mathbb{R}$ satisfying

(i) $d(a, b) = 0$ if and only if $a = b$.

(ii) $d(a, b) + d(a, c) \geq d(b, c)$ for all $a, b, c \in A$.

(iii) $d(a, b) \geq 0$ for all $a, b \in A$.

(iv) $d(a, b) = d(b, a)$ for all $a, b \in A$.

Consider also

(i)$'$ $d(a, a) = 0$ for all $a \in A$.

Show that (i) and (ii) taken together imply (iii) and (iv). Do conditions (i)$'$ and (ii) taken together imply either of (iii) or (iv)?

2. Each of the following functions $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies (i) and (ii), so is a metric. For each of them, decide whether 'it is not difficult to see' that this is so:

(a) the *usual metric* $d(x, y) = (\sum_{i=1}^n (x_i - y_i)^2)^{\frac{1}{2}}$ is 'not difficult'? Yes No

(b) $d(x, y) = \sum_{i=1}^n |x_i - y_i|$ Yes No

(c) The *discrete metric*: $d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$ Yes No

(d) $d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$. Yes No

3.* Show that $d(x, y) = (x - y)^2$ does not define a metrix on $\mathbb{R}$.

4. Show that $d(x, y) = \min_{1 \leq i \leq n} |x_i - y_i|$ does not define a metrix on $\mathbb{R}^n$.

5. On $\mathbb{R}^2$ define $d(x, y) =$ smallest integer greater or equal to usual distance between x and y. Is d a metric for $\mathbb{R}^2$?

6.* Let A be a metric space with metric d. Let $y \in A$ be any point. Show that the function $f : A \to \mathbb{R}$ defined by $f(x) = d(x, y)$ is continuous, where $\mathbb{R}$ has the usual metric.

7. Let M be the metric space $(\mathbb{R}, d)$ where d is the usual Euclidean metric. Let M_0 be the metric space $(\mathbb{R}, d_0)$ where d_0 is the discrete metric. Show that all functions $f : M_0 \to M$ are continuous. Show that there does not exist any injective continuous function from M to M_0.

8. Let A, B be metric spaces with metrics d and d_B respectively. For each number $r > 0$ let d_r be the metric on A given by $d_r(x, y) = rd(x, y)$. Let f be a function from A to B. Prove that f is continuous with respect to the metric d on A if and only if it is continuous with respect to the metric d_r on A.

9.* Show that $B_\epsilon(x)$ is always an open set for all x and all $\epsilon > 0$.

10. Which of the following subsets of $\mathbb{R}^2$ (with the usual topology) are open?

a. $\{(x, y) \mid x^2 + y^2 < 1\} \cup \{(1, 0)\}$,

b. $\{(x, y) \mid x^2 + y^2 \leq 1\}$

c. $\{(x, y) \mid |x| < 1\}$,

d. $\{(x, y) \mid x + y < 0\}$,

e. $\{(x, y) \mid x + y \geq 0\}$,

f. $\{(x, y) \mid x + y = 0\}$.

11(a). Show that if $\mathcal{F}$ is the family of open sets arising from a metric space then

(i) The empty set $\emptyset$ and the whole set belong to $\mathcal{F}$.

(ii) The intersection of two members of $\mathcal{F}$ belongs to $\mathcal{F}$,

(iii) The union of *any* number of members of $\mathcal{F}$ belongs to $\mathcal{F}$.

11(b). Give an example of an infinite collection of open sets of $\mathbb{R}$ (with the usual metric) whose intersection is not open.

12(a).* Let d_1 and d_2 be two metrics on the set A, giving open balls denoted $B_{1,\epsilon}(x)$ and $B_{2,\epsilon}(x)$. Show that the open sets for metric d_1 are also open for the metric d_2 if and only if the balls $B_{1,\epsilon}(x)$ are open for metric d_2.

12(b).* Let $A = \mathbb{R}^2$ and let

$$d_0(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\},$$
$$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|,$$
$$d_2 = \text{the usual metric.}$$

Draw pictures of (or otherwise describe) the unit balls centered on the origin for each of the metrics.

12(c).* Decide whether or not d_1, d_2, d_3 have the same open sets on $\mathbb{R}^2$.