Math 5345H **Worksheet 3** **9/8/2025**

Upload to Gradescope the starred questions 2c, 6abc, 7c, 8deh from Worksheet 3 before the end of the day on Wednesday 9/24/2025

1. (a) Let X be a topological space that is metrizable. Prove that for every pair a, b of distinct points of X there are open sets U_a and U_b containing a and b respectively, such that $U_a \cap U_b = \emptyset$.
(b) Show that if X has finitely many points and is metrizable then X has the discrete topology. Give an example of a topological space that is not metrizable.

2. In each case (a), (b), (c) below show that $\mathcal{U}$ is a topology for X.
(a) $X = \mathbb{R}$ and $\mathcal{U} = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, x) \mid x \in \mathbb{R}\}$.
(b) $X = \mathbb{N}$ and $\mathcal{U} = \{\emptyset, \mathbb{N}\} \cup \{O_n \mid n \geq 1\}$ where $O_n = \{n, n+1, n+2, \ldots\}$.
(c)* $X = \mathbb{R}$ and $U \in \mathcal{U}$ if and only if U is a subset of $\mathbb{R}$ and for each $s \in U$ there is a $t > s$ such that the half-open interval $[s, t) \subseteq U$.
(d) Determine the number of distinct topologies on a set with three elements.
(e) Show the neither of the following families of subsets of $\mathbb{R}$ is a topology.

$$\mathcal{U}_1 = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, x] \mid x \in \mathbb{R}\},$$
$$\mathcal{U}_2 = \{\emptyset, \mathbb{R}\} \cup \{(a, b) \mid a, b \in \mathbb{R},\ a < b\},$$

3. Is the following obvious? Yes / No

Theorem. *Let X be a set and let $\mathcal{V}$ be a family of subsets of X satisfying*
(i) $\emptyset, X \in \mathcal{V}$,
(ii) the union of any pair of elements of $\mathcal{V}$ belongs to $\mathcal{V}$,
(iii) the intersection of any number of elements of $\mathcal{V}$ belongs to $\mathcal{V}$.
Then $\mathcal{U} = \{X - V \mid V \in \mathcal{V}\}$ is a topology for X.

4. Is the following obvious? Yes / No

Theorem. *In a discrete topological space each subset is simultaneously open and closed.*

5. Is the following obvious? Yes / No

Theorem. *If a topological space has only a finite number of points each of which is closed then it has the discrete topology.*

6.* Consider the topological space $X = (\mathbb{R}, \mathcal{U})$, where $\mathcal{U}$ is as defined in 2(c).
(a)* Show that each of the sets $[s, t)$ is both an open and a closed subset.
(b)* Find the closure and interior of each of the following subsets of X:

$$(a, b), [a, b), (a, b], [a, b].$$

(c)* True or false?: The topology $\mathcal{U}$ contains the usual topology on $\mathbb{R}$. (In other words, $\mathcal{U}$ is *finer* than the usual topology on $\mathbb{R}$.)

7. The boundary of a subset Y of a topological space X is defined to be

$$\partial Y := \overline{Y} - Y^\circ = \overline{Y} \cap \overline{(X - Y)}.$$

(a) Let X be $\mathbb{R}$ with its usual topology. Find the closure, interior and boundary of each of the following subsets of X:

$$A = \{1, 2, 3, \ldots\}, \quad B = \{x \mid x \text{ is rational}\}, \quad C = \{x \mid x \text{ is irrational}\}.$$

(b) Let X be $\mathbb{R}^2$ with its usual topology. Find the closure, interior and boundary of each of the following subsets of X: the circle $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$; the disc $D^2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$; the line $\{(x, y) \mid x = y\}$.
(c)* Let $X = \{a, b, c\}$ with the topology whose open sets are $\emptyset, \{a\}, \{c\}, \{a, c\}$ and X. For each of the following sets find the closure, interior and boundary.
(i) $\{a, b\}$,
(ii) $\{a\}$.

8. Prove each of the following statements for a subset Y of a topological space X.
(a) If F is a subset of X with $Y \subseteq F \subseteq X$ and F is closed then $\overline{Y} \subseteq F$.
(b) Y is closed if and only if $Y = \overline{Y}$.
(c) $\overline{\overline{Y}} = \overline{Y}$.
(d)* $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Give an example where the second containment fails to be an equality.
(e)* $X - Y^\circ = \overline{X - Y}$.
(f) $\overline{Y} = Y \cup \partial Y$.
(g) Y is closed if and only if $\partial Y \subseteq Y$.
(h)* $\partial Y = \emptyset$ if and only if Y is both open and closed.
(i) $\partial((a, b)) = \partial([a, b]) = \{a, b\}$ where (a, b) and $[a, b]$ denote the open and closed intervals in $\mathbb{R}$ with the usual topology.
(j) Prove that Y is the closure of some open set if and only if Y is the closure of its interior.