Math 5345H **Worksheet 4** **9/22/2025**

Upload to Gradescope the starred questions 4, 6acde, 7ab from Worksheet 4 before the end of the day on Wednesday 10/1/2025

1. Are the following obvious?

(a) Let X be an arbitrary set and let $\mathcal{U},\mathcal{U}'$ be topologies on X. Prove that the identity mapping $(X,\mathcal{U}) \to (X,\mathcal{U}')$ is continuous if and only if $\mathcal{U}' \subseteq \mathcal{U}$. (Or, should it be $\mathcal{U} \subseteq \mathcal{U}'$?)

(b) X is a topological space with the property that, for every topological space Y, every function $f : X \to Y$ is continuous. Prove that X has the discrete topology. (Hint: Let Y be the space X but with the discrete topology.)

(c) Y is a topological space with the property that, for every topological space X, every function $f : X \to Y$ is continuous. Prove that Y has the indiscrete topology. (Hint: Let X be the space Y but with the indiscrete topology.)

2. A *partially ordered set* P (or *poset*) is a set with a reflexive, transitive and skew-symmetric relation, written $x \leq y$. These three conditions are
(i) $x \leq x$ for all $x \in P$,
(ii) $x \leq y$ and $y \leq z$ implies $x \leq z$
(iii) $x \leq y$ and $y \leq x$ implies $x = y$.
Often we draw a picture of P as a graph with the elements of P as vertices and with x lower than y with an edge between x and y if $x \leq y$. If we only draw such edges when there is no further z with $x < z < y$ we call this picture the *Hasse diagram* of P.

(a) Draw pictures of all the poset structures on a 2-element set $P = \{a, b\}$, up to relabeling the elements.

(b) Draw pictures of all the poset structures on a 3-element set $P = \{a, b, c\}$, up to relabeling the elements.

3. Given a finite poset P, let $\mathcal{U}$ be the collection of subsets U of P satisfying the condition
- $x \in U$ and $y \leq x$ implies $y \in U$.

(a) Show that $\mathcal{U}$ is a topology on P. (There are other ways to make P into a topological space that we do not study here).

(b) For each of the poset structures on the 3-element set $P = \{a, b, c\}$ (up to relabeling the elements) find how many open sets the corresponding topology has.

(c) Find a topology on $\{a, b, c\}$ that does not come from any poset structure on these three elements.

4.* A map of sets $f : P \to Q$ between posets P and Q is said to be *order preserving* if whenever $x \leq y$ in P we have $f(x) \leq f(y)$ in Q. Show that such an order preserving map gives rise to a continuous map between the topological spaces P and Q, with the same definition of f.

5. Let $X = \{a, b, c\}$ have the topology with open sets $\emptyset, \{a\}, \{a, b\}, \{a, b, c\}$ and let $Y = \{a, b, \}$ have the topology with open sets $\emptyset, \{a\}, \{a, b\}$.

(a) Find the interior, closure and boundary in X of the set $\{a\}$; and also of the set $\{b\}$. Do the topologies on X and Y come from posets?

(b) Let $f : X \to Y$ be the mapping $f(a) = a, f(b) = b, f(c) = b$. Find if f is continuous; open; closed.

(c) Let $f : X \to Y$ be the mapping $f(a) = a, f(b) = b, f(c) = a$. Find if f is continuous; open; closed.

6. Let $X = \{a, b, c\}$ and consider the collection of subsets $\mathcal{B} = \{\{a\}, \{a, b\}, \{c\}\}$.

(a)* Show that $\mathcal{B}$ is a basis for a topology on X. Show that the open sets in this topology are $\emptyset, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}$.

(b) Does this topology on X come from a poset?

(c)* Find the interior, closure and boundary in X of the set $\{b, c\}$.

(d)* Let $Y = \{a, b, \}$ have the topology with open sets $\emptyset, \{a\}, \{a, b\}$, and let $f : X \to Y$ be the mapping $f(a) = a, f(b) = b, f(c) = b$. Find if f is continuous; open; closed.

(e)* Find a continuous, surjective map between topological spaces that is open, but not closed.

7. (a)* For any subset W of a topological space X, show that the boundary ∂W is a closed set.

(b)* Show that a subset U of a topological space X is open if and only if U contains no point of its boundary.

8. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for two potentially different topologies $\mathcal{U}_1$ and $\mathcal{U}_2$ on a single set X. Show that the topologies are the same if and only if the sets in $\mathcal{B}_1$ are all open in $\mathcal{U}_2$, and the sets in $\mathcal{B}_2$ are all open in $\mathcal{U}_1$.