Math 5345H **Worksheet 6** **10/05/2025**

Upload to Gradescope the starred questions 4a, 5 from Worksheet 6 before the end of the day on Wednesday 10/17/2025

1. Let $\mathbb{R}$ have its usual topology and let $\sim$ be the equivalence relation on $\mathbb{R}$ given by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Identify the quotient topology on $\mathbb{R}/\sim$. (Is it a topology you recognize? Does it have a name?) Is the quotient map $\mathbb{R} \to \mathbb{R}/\sim$ open or closed (or neither)?

2.(a) We can define the circle S^1 as $\mathbb{R}/\sim$ where $\sim$ is the equivalence relation $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Show that the quotient map $\mathbb{R} \to \mathbb{R}/\sim$ is open but not closed. Here $\mathbb{R}$ has its usual topology.
(b) We can also define the circle S^1 as $[0,1]/\sim$ where $\sim$ is the equivalence relation $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Show that the quotient map $\mathbb{R} \to \mathbb{R}/\sim$ is closed but not open. Here $[0,1]$ is the closed interval that is a subset of $\mathbb{R}$, with its usual topology.

3. Show that if we have two pairs of homeomorphic spaces $X_1 \cong X_2$ and $Y_1 \cong Y_2$ then $X_1 \times Y_1 \cong X_2 \times Y_2$.

4. Let X, Y be metrizable spaces and suppose that they arise from metrics d_X, d_Y respectively.
(a)* Show that d defined by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

is a metric on $X \times Y$ that produces the product space topology on $X \times Y$. (Hint: use the result of question 8 on Worksheet 4.)
(b) Deduce that the product topology on $\mathbb{R}^m \times \mathbb{R}^n$ (where $\mathbb{R}^m$ and $\mathbb{R}^n$ have the usual topology) is the same as the usual topology on $\mathbb{R}^{m+n}$ when $\mathbb{R}^{m+n}$ is identified with $\mathbb{R}^m \times \mathbb{R}^n$ in the usual way.

5.* The *graph* of a function $f : X \to Y$ is the set of points in $X \times Y$ of the form $(x, f(x))$ for $x \in X$. Given such a mapping, let $F : X \to X \times Y$ be the mapping $F(x) = (x, f(x))$. Let X, Y be topological spaces.
(a)* Show that if f is a continuous then F is a homeomorphism between X and the graph of f. (Note that the topology on the graph of f has not been specified, but are you in any doubt as to what it should be?)
(b)* Show that if F is a homeomorphism between X and the graph of f then f is continuous.

6. Prove that $\mathbb{R}^2 - \{0\}$ is homeomorphic to $\mathbb{R} \times S^1$. (Hint: Consider $\mathbb{R}^2 - \{0\}$ as $\mathbb{C} - \{0\}$ and the $re^{i\theta}$ expression for complex numbers.)

7. Given two topological spaces X, Y consider topological spaces Z satisfying the two properties:
(i) there are continuous maps $\iota_X : X \to Z$ and $\iota_Y : Y \to Z$, so that
(ii) given any pair of continuous maps $f : X \to W$ and $g : Y \to W$ there exists a unique continuous map $h : Z \to W$ so that $f = h\iota_X$ and $g = h\iota_Y$.

(a) Does it seem reasonable that the product $Z = X \times Y$ has properties (i) and (ii)?
(b) Show that the disjoint union $Z = X \sqcup Y$ has properties (i) and (ii).
(c) Harder: show that any space satisfying (i) and (ii) is uniquely determined up to homeomorphism by these properties.

8. Show that the product topology on $X \times Y$ is the weakest topology such that the maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous.

9. Prove that the torus (defined by identifying opposite edges of a square so that the two vertical edges are both oriented bottom to top and the two horizontal edges are both oriented left to right) is homeomorphic to $S^1 \times S^1$.