Math 5345H **Worksheet 7** **10/12/2025**

Upload to Gradescope the starred questions 3, 11, 12 from Worksheet 7 before the end of the day on Wednesday 10/22/2025

1. Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ given by

$$(x_1, x_2) \sim (y_1, y_2) \Leftrightarrow x_2 = y_2.$$

Show that the identification space $\mathbb{R}^2/\sim$ is homeomorphic to $\mathbb{R}$.

2. Let A be a subset of a topological space X, and let B be a subset of a topological space Y. Show that in the product space $X \times Y$ we have $(A \times B)^\circ = A^\circ \times B^\circ$.

3.* Suppose that X has the finite complement topology. Show that X is compact; moreover, every subset of X is compact.

4. Show that the following are equivalent for a topological space X:
(a) X is compact.
(b) Whenever $\{C_j \mid j \in J\}$ is a collection of closed sets with $\bigcap_{j\in J} C_j = \emptyset$ then there is a finite subcollection $\{C_k \mid k \in K\}$ such that $\bigcap_{k\in K} C_k = \emptyset$.
(c) Whenever $\{C_j \mid j \in J\}$ is a collection of closed sets with the property that $\bigcap_{k\in K} C_k \neq \emptyset$ for every finite subset K of J, then $\bigcap_{j\in J} C_j \neq \emptyset$.
[A collection of sets with the property that intersections of finitely many of them are always non-empty is said to have the Finite Intersection Property. Thus (c) says that every collection of closed subsets with the FIP has non-empty intersection.]

5. Consider whether you are able to prove the following theorem by modifying the proof given in Kosniowski's book of Theorem 7.7 (it says: $[0, 1]$ is compact).

Theorem. *(a) Every infinite sequence of points in* $[0, 1]$ *has a convergent subsequence. (b) Every infinite subset of* $[0, 1]$ *has a limit point (whatever that is!).*

6. Let $\mathcal{T}$ be the topology on $\mathbb{R}$ generated by the basis $\{[s, t) \mid s < t\}$. (Equivalently $U \in \mathcal{T}$ if and only if for each $s \in U$ there is a $t > s$ such that $[s, t) \subseteq U$.) Prove that the subset $[0, 1]$ of $(\mathbb{R}, \mathcal{T})$ is not compact.

7. Is it obvious that the following result in analysis is an immediate consequence of results in Ch. 7 on compact spaces?

Theorem. *Let* $f : [0, 1] \to \mathbb{R}$ *be a continuous function. Then* f *is bounded and achieves its bounds.*

8. Which of the following spaces are compact? Why?
$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$,
$\{x \in \mathbb{R}^n \mid \|x\| < 1\}$,
$\{(s, t) \in \mathbb{R}^2 \mid 0 \leq s \leq 1,\ 0 \leq t \leq 4\}$,
$\{(s, t, u) \in \mathbb{R}^3 \mid s^2 + t^2 \leq 1\} \cap \{(s, t, u) \in \mathbb{R}^3 \mid t^2 + u^2 \leq 1\}$.

9. Prove that a compact subset of $\mathbb{R}^n$ is bounded.

10. Prove that the graph of a function $f : [0,1] \to \mathbb{R}$ is compact if and only if f is continuous. Give an example of a discontinuous function $g : [0,1] \to \mathbb{R}$ with a graph that is closed but not compact.

11.* Let X be a compact topological space arising from some metric space with metric d. Prove that if $\{U_j \mid j \in J\}$ is an open cover of X then there exists a real number $\delta > 0$ (called the *Lebesgue number* of $\{U_j \mid j \in J\}$) such that any subset of X of diameter less than δ is contained in one of the sets U_j, $j \in J$. [Hint: create another cover using balls, and use balls half the size you might first have thought.]

12.* Let X be a topological space and define X^∞ to be $X \cup \{\infty\}$ where ∞ is some new element not contained in X. If $\mathcal{T}$ is the topology for X then define $\mathcal{T}^\infty$ to be $\mathcal{T}$ together with all sets of the form $V \cup \{\infty\}$ where $V \subseteq X$ and $X - V$ is both compact and closed in X. Prove that $\mathcal{T}^\infty$ is a topology for X^∞ and that X^∞ is compact. (X^∞ is called the *one-point compactification* of X.)