Math 5345H **Worksheet 8** **10/19/2025**

Upload to Gradescope the starred questions 4, 7, 8, 11 from Worksheet 8 before the end of the day on Wednesday 10/29/2025

1. Let X be a space with the finite complement topology. Prove that X is Hausdorff if and only if X is finite.

2. Let $\mathcal{T}$ be the topology on $\mathbb{R}$ generated by the basis $\{[s,t) \mid s < t\}$. (Equivalently $U \in \mathcal{T}$ if and only if for each $s \in U$ there is a $t > s$ such that $[s,t) \subseteq U$.) Prove that $(\mathbb{R}, \mathcal{T})$ is Hausdorff.

3. Construct topological spaces X_0, X_1, X_2 and X_3 with the property that X_k is a T_k-space, but X_k is not a T_j-space for $j > k$.

4.* Show that a finite T_1-space must be Hausdorff ($= T_2$).

5. (= qn. 7 from Worksheet 6) Is it obvious that the following result in analysis is an immediate consequence of results in Ch. 7 on compact spaces?

Theorem. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Then f is bounded and achieves its bounds.*

6. Let $f : X \to Y$ be a continuous surjective map of a compact space X onto a Hausdorff space Y. Prove that a subset U of Y is open if and only if $f^{-1}(U)$ is open in X. (Hint: Prove that a subset C of Y is closed if and only if $f^{-1}(C)$ is closed in X.) Deduce that Y has the quotient topology determined by f.

7.* Prove that the space Y is Hausdorff if and only if the diagonal

$$D = \{(y_1, y_2) \in Y \times Y \mid y_1 = y_2\}$$

in $Y \times Y$ is a closed subset of $Y \times Y$.

8.* Let $f : X \to Y$ be a continuous map. Prove that if Y is Hausdorff then the set $\{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$ is a closed subset of $X \times X$.

9. Let $f : X \to Y$ be a map that is continuous, open and onto. Prove that Y is a Hausdorff space if and only if the set $\{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$ is a closed subset of $X \times X$.

10. Let X be a compact Hausdorff space and let Y be a quotient space determined by a map $f : X \to Y$. Prove that Y is Hausdorff if and only if f is a closed map. Furthermore, prove that Y is a Hausdorff space if and only if the set $\{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$ is a closed subset of $X \times X$.

11.* Take $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Regard the closed interval $[0,1]$ as being part of the real axis in $\mathbb{C}$. Let $\sim$ be the equivalence relation on $S^1 \times [0,1]$ given by $(x,t) \sim (y,s)$ if and only if $xt = ys$. This means the only non-trivial identification given by $\sim$ is when $s = t = 0$, so that $(x,0) \sim (y,0)$ for all $x, y \in S^1$. Prove that $(S^1 \times [0,1])/\sim$ is homeomorphic to the unit disc $D^2 = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\} = \{x \in \mathbb{C} \mid |x| \leq 1\}$ with the induced topology.

12. Let $\sim$ be the equivalence relation on the unit square X in $\mathbb{R}^2$ that identifies each point on its boundary with the diametrically opposite point. Prove that $X/\sim$ is homeomorphic to $\mathbb{R}P^2$.

13. Let X be a compact Hausdorff space and let U be an open subset of X not equal to X itself. The one-point compactification X^∞ of X is defined in question 12 of Worksheet 7, which is question 7.13(h) of Kosniowski's book. Prove that $U^\infty \cong X/(X-U)$.

(Hint: Consider $h : U^\infty \to X/(X-U)$ given by $h(u) = p(u)$ for $u \in U$ and $h(\infty) = p(X-U)$ where $p : X \to X/(X-U)$ is the natural projection.)

Deduce that if $x \in X$ (and X is a compact Hausdorff space) then $(X - \{x\})^\infty \cong X$.

14. Prove that

$$S^n \cong (\mathbb{R}^n)^\infty \cong D^n/S^{n-1} \cong I^n/\partial I^n$$

where D^n is the closed unit ball and I^n is the product of n copies of the unit interval in $\mathbb{R}^n$.

15. Make sure you are familiar with the steps in the proof that: *A subset of* $\mathbb{R}^n$ *is compact if and only if it is closed and bounded.*