Math 5345H **Worksheet 9** **10/26/2025**

Upload to Gradescope the starred questions 4, 8, 13 from Worksheet 9 before the end of the day on Wednesday 11/5/2025

1. Prove that the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ is not a connected set. What are the connected subsets of $\mathbb{Q}$?

2. How obvious are the following for a space X?

(i) If X has the discrete topology then the only connected subsets of X are single point subsets.

(ii) If X has the indiscrete topology then every subset of X is connected.

3. Which of the following subsets of $\mathbb{R}^2$ are connected?

$$\{x \mid \|x\| < 1\}, \quad \{x \mid \|x\| > 1\}, \quad \{x \mid \|x\| \neq 1\}.$$

4.* Let X be a set with 3 elements: $X = \{a, b, c\}$. In each part, X is given a different topology.

(i) In the topology $\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$, is X connected?

(ii) In the topology $\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$, is X connected?

(iii) In the topology $\mathcal{T}_3 = \{\emptyset, X, \{a\}, \{a, b\}, \{c\}\{a, c\}\}$, is X connected?

5. Prove that a subset of $\mathbb{R}$ is connected if and only if it is an interval or a single point. (A subset S of $\mathbb{R}$ is called an *interval* if S contains at least two distinct points, and if $a, b \in S$ with $a < b$ and $a < x < b$ then $x \in S$.)

6. Show that the 6 spaces that are the intervals $(0, 1), (0, 1], [0, 1]$, the circle S^1, 2-dimensional space $\mathbb{R}^2$, and the union of 3 copies of $[0, 1]$ all identified together at a single point 1 (forming a Y) are all non-homeomorphic.

7. Prove that a topological space X is connected if and only if each continuous mapping of X into a discrete space (with at least two points) is a constant mapping.

8.* Let A be a connected subspace of X with $A \subseteq Y \subseteq \overline{A}$. Prove that Y is connected.

9. Suppose that Y_0 and $\{Y_j \mid j \in J\}$ are connected subsets of a space X. Prove that if $Y_0 \cap Y_j \neq \emptyset$ for all $j \in J$ then $Y = Y_0 \cup (\bigcup_{j \in J} Y_j)$ is connected.

10. Prove that $\mathbb{R}^{n+1} - \{0\}$ is connected if $n \geq 1$. Deduce that S^n and $\mathbb{R}P^n$ are connected for $n \geq 1$. (Hint: Assume question 8. Consider $f : (\mathbb{R}^{n+1} - \{0\}) \to S^n$ given by $f(x) = x/\|x\|$.)

11. This space and the space in the next question are the kind of weird things that are often considered at this point in this kind of course. Let A and B be subsets of $\mathbb{R}^2$ defined by

$$A = \{(x, y) \mid x = 0, \ -1 \leq y \leq 1\}$$
$$B = \{(x, y) \mid 0 \leq x \leq 1, \ y = \cos(\pi/x)\}$$

Prove that $X = A \cup B$ is connected. (Draw a picture. Prove that A and B are connected. If $X = U \cup V$ where U, V are open and closed in X, figure out what they must be. Finally assume that some point of A is in U.)

12. Let A and B be subsets of $\mathbb{R}^2$ defined by

$$A = \{(x, y) \mid 1/2 \leq x \leq 1,\ y = 0\}$$
$$B = \{(x, y) \mid 0 \leq x \leq 1,\ y = x/n \text{ where } n \in \mathbb{N}\}$$

Prove that $X = A \cup B$ is connected.

13.* Show that if X is an infinite set, it is connected in the finite complement topology.

14. Define a relation on a topological space X by $x \sim y$ if and only if there is a connected subspace of X containing both x and y.

(i) Show that $\sim$ is an equivalence relation. (The equivalence classes are called the *connected components* of X.)

(ii) What are the connected components of the spaces listed in questions 1 and 4?

(iii) Show that (in general, not just for those questions) the connected components are a) connected and b) closed. Show by example that the connected components need not be open.