Homework Assignment 2 Due Thursday 10/21/2021, uploaded to Gradescope.

1. (2.10 of Eisenbud) Let \( R \) be a commutative ring. Show that every finitely generated module over \( R[U^{-1}] \) is the localization of a finitely generated module over \( R \). (Eisenbud also notes that the same implication without the condition finitely generated looks deeper but is a triviality. Do not address this comment.)

2. Let \( A = M_{m,m}(R) \) and \( B = M_{n,n}(R) \) be matrix rings over a commutative ring \( R \). Show that \( A \otimes_R B \cong M_{mn,mn}(R) \), where the multiplication giving the ring structure on the tensor product is determined by \((a \otimes b)(c \otimes d) := ac \otimes bd\) as in class and on page 65 of Eisenbud.

3. If \( R \) is any integral domain with quotient field \( Q \), prove that \((Q/R) \otimes_R (Q/R) = 0\).

4. Let \( \{e_1, e_2\} \) be a basis of \( V = \mathbb{R}^2 \). Show that the element \( e_1 \otimes e_2 + e_2 \otimes e_1 \) in \( V \otimes_\mathbb{R} V \) cannot be written as a simple tensor \( v \otimes w \) for any \( v, w \in \mathbb{R}^2 \).

5. (a) Let \( K \supseteq \mathbb{Q} \) be a field containing \( \mathbb{Q} \). Show that \( K \otimes_\mathbb{Q} \mathbb{Q}[x] \cong K[x] \) as rings
(b) Show that, as a ring, \( \mathbb{Q}(\sqrt{2}) \otimes_\mathbb{Q} \mathbb{Q}(\sqrt{2}) \) is the direct sum of two fields.
[The ring multiplication is \((a \otimes b)(c \otimes d) := ac \otimes bd\) on basic tensors. Use the isomorphism \( \mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}[x]/(x^2 - 2) \).]

6. Let \( 0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0 \) be a short exact sequence of \( R \)-modules, for some ring \( R \). Suppose that \( A \) can be generated as an \( R \)-module by a subset \( X \subseteq A \) and that \( C \) can be generated as an \( R \)-module by a subset \( Y \subseteq C \). For each \( y \in Y \), choose \( y' \in B \) with \( \beta(y') = y \). Prove that \( B \) is generated by the set \( \alpha(X) \cup \{y' \mid y \in Y\} \).

7. Let \( A, B \) be left \( R \)-modules and let \( r \in Z(R) = \{s \in R \mid st = ts \text{ for all } t \in R\} \), the center of \( R \). Let \( \mu_r : B \rightarrow B \) be multiplication by \( r \). Prove that the induced map \( (\mu_r)_* : \text{Hom}_R(A, B) \rightarrow \text{Hom}_R(A, B) \) is also multiplication by \( r \).

Extra questions: do not upload to Gradescope.

8. Let \( A \) be a finite abelian group of order \( n \) and let \( p^k \) be the largest power of the prime \( p \) dividing \( n \). Prove that \( \mathbb{Z}/p^k\mathbb{Z} \otimes_\mathbb{Z} A \) is isomorphic to the Sylow \( p \)-subgroup of \( A \).

9. (Part of 2.4 of Eisenbud) Let \( k \) be a field and let \( m, n \) be integers. Describe as explicitly as possible the following. (For example, if the object is a finite-dimensional vector space, what is its dimension?)
   a. \( \text{Hom}_{k[x]}(k[x]/(x^n), k[x]/(x^m)) \)
   b. \( k[x]/(x^n), \otimes_{k[x]} k[x]/(x^m) \)
c. \( k[x] \otimes_k k[x] \) (describe this as an algebra).

10. Eisenbud question 2.11 (It is very similar for modules to something we did for rings.

11. Let \( U \subset R \) be a multiplicative subset not containing any zero divisors of a commutative ring \( R \). We can regard \( R \) as the set of elements \( \frac{r}{1} \) of \( R[U^{-1}] \) where \( r \) ranges through \( R \). If \( S \) is a ring with \( R \subseteq S \subseteq R[U^{-1}] \), show that \( S[U^{-1}] = R[U^{-1}] \).

12. Suppose that \( U \) and \( V \) are two multiplicative subsets of the commutative ring \( R \) with \( U \subseteq V \). Writing \( V' \) for the image of \( V \) in \( R[U^{-1}] \), show that \( R[U^{-1}][V'^{-1}] = R[V^{-1}] \).