Homework Assignment 2 Due Saturday 3/5/2022, uploaded to Gradescope.
Each question part is worth 1 point. There are 12 question parts. Assume that all categories are small. We define Fun(\mathcal{C}, \mathcal{D}) to be the category whose objects are functors \mathcal{C} \to \mathcal{D} and whose morphisms are natural transformations.

1. Suppose that \( F : \mathcal{C} \to \mathcal{D} \) is an equivalence of categories.
   (a) Show that, for all objects \( x, y \in \text{Ob} \mathcal{C} \), the functor \( F \) provides a bijection
   \[
   \text{Hom}_\mathcal{C}(x, y) \leftrightarrow \text{Hom}_\mathcal{D}(F(x), F(y)),
   \]
   that preserves composition, so that \( \text{End}_\mathcal{C}(x) \cong \text{End}_\mathcal{D}(F(x)) \) as monoids.
   (b) Show that \( x \cong y \) in \( \mathcal{C} \) if and only if \( F(x) \cong F(y) \) in \( \mathcal{D} \), so that \( F \) provides a bijection between the isomorphism classes of \( \mathcal{C} \), and of \( \mathcal{D} \).
   (c) Let \( \mathcal{E} \) be a further category. Show that the functor categories Fun(\mathcal{C}, \mathcal{E}) and Fun(\mathcal{D}, \mathcal{E}) are naturally equivalent.

2. Let \( \mathcal{C} \) be a category and let \( x, y \in \text{Ob} \mathcal{C} \). Prove that if \( x \cong y \) then \( \text{Hom}_\mathcal{C}(x, -) \) and \( \text{Hom}_\mathcal{C}(y, -) \) are naturally isomorphic functors \( \mathcal{C} \to \text{Set} \).

3. Let \( F, G : \mathcal{C} \to \mathcal{D} \) be functors and \( \eta : F \to G \) a natural transformation.
   (a) Show that if, for all \( x \in \text{Ob} \mathcal{C} \), the mapping \( \eta_x : F(x) \to G(x) \) is an isomorphism in \( \mathcal{D} \), then \( \eta \) is a natural isomorphism (meaning that it has a 2-sided inverse natural transformation \( \theta : G \to F \)).
   (b) Suppose that \( F \) is an equivalence of categories and that \( F \) is naturally isomorphic to \( G \), so \( F \cong G \). Show that \( G \) is an equivalence of categories.

4. Let \( G \) be a group, which we regard as a category \( \mathcal{G} \) with a single object, and with the elements of \( G \) as morphisms. Let \( F : \mathcal{G} \to \mathcal{G} \) be a functor.
   (a) Show that \( F \) is naturally isomorphic to the identity functor \( 1_\mathcal{G} : \mathcal{G} \to \mathcal{G} \) if and only if the mapping \( F : G \to G \), induced by \( F \) on the set of morphisms, is an inner automorphism; that is, an automorphism of the form \( c_g : G \to G \) for some \( g \in G \), where \( c_g(h) = ghg^{-1} \) for all \( h \in G \).
   (b) Show that self equivalences of \( \mathcal{G} \) are automorphisms of \( \mathcal{G} \).
   (c) Show that the group of natural isomorphism classes of self equivalences of \( \mathcal{G} \) is isomorphic to \( \text{Aut}(G)/\text{Inn}(G) \). (In the context of group theory, \( \text{Inn}(G) \) denotes the set of inner automorphisms of \( G \), and \( \text{Out}(G) := \text{Aut}(G)/\text{Inn}(G) \) is called the group of outer (or non-inner) automorphisms.)
5. Let $I$ be the poset with two elements $0$ and $1$, and with $0 < 1$. If $P$ and $Q$ are posets we can regard them as categories $\mathcal{P}$ and $\mathcal{Q}$ whose objects are the elements of the posets, and where there is a unique morphism $x \to y$ if and only if $x \leq y$.

(a) Show that if $P$ and $Q$ are posets then a functor $P \to Q$ is ‘the same thing as’ an order-preserving map. (Don’t worry about any fancy interpretation of ‘the same thing as’!)

(b) Now consider two functors $F, G : \mathcal{P} \to \mathcal{Q}$, which we may regard as order-preserving maps $f, g : P \to Q$ by part (a). Show that the following three conditions are equivalent:

(i) there exists a natural transformation $F \to G$,

(ii) $f(x) \leq g(x)$ for all $x \in P$,

(iii) there is an order-preserving map $h : P \times I \to Q$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in \mathcal{P}$. Here $P \times I$ denotes the product poset with order relation $(a_1, b_1) \leq (a_2, b_2)$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$, where $a_i \in P$ and $b_i \in I$.

6. Let $1_{R\text{-mod}} : R\text{-mod} \to R\text{-mod}$ denote the identity functor. Let $\text{Nat}(1_{R\text{-mod}}, 1_{R\text{-mod}})$ denote the set of natural transformations from this functor to itself, noting that this set has the structure of a ring (multiplication is composition and addition comes because we can add homomorphisms of $R$-modules, so that for two natural transformations $\theta, \psi$ at an object $x$ we have $(\theta + \psi)_x = \theta_x + \psi_x$). Show that $\text{Nat}(1_{R\text{-mod}}, 1_{R\text{-mod}}) \cong \mathbb{Z}(R)$.

Extra question: do not upload to Gradescope.

7. Let $\mathcal{C}$ be a small category and let $F, G : \mathcal{C} \to \text{Set}$ be functors. Show that a natural transformation of functors $\tau : F \to G$ is an epimorphism in $\text{Fun}(\mathcal{C}, \text{Set})$ if and only if for every object $x$ of $\mathcal{C}$, $\tau_x : F(x) \to G(x)$ is a surjection; and it is a monomorphism if and only if for every object $x$ of $\mathcal{C}$, $\tau_x : F(x) \to G(x)$ is a 1-1 map.

8. Write out a proof that if $G$ is the right adjoint of a functor $F$ with the property that $F$ preserves monomorphisms, then $G$ sends injective objects to injective objects.

9. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors with $F$ left adjoint to $G$, and with adjunction unit $\eta$ and counit $\epsilon$. Write out a proof that the second triangular identity holds, namely the following triangle commutes:

$$
\begin{array}{ccc}
\text{G} & \xrightarrow{1_{\mathcal{C}}} & \text{G} \\
\downarrow{\eta_{\text{G}}} & & \downarrow{\text{G}\epsilon} \\
\text{GFG} & & \\
\end{array}
$$

10. Assume the axiom of choice in this question, or else make some assumption such as: everything is finite. Let $\mathcal{C}$ be a category, and for each isomorphism class $\hat{x}$ of objects $x$, choose a fixed representative $u_{\hat{x}}$. For each object $x$ choose a fixed isomorphism $i_x : x \to u_{\hat{x}}$. Let $\mathcal{D}$ be the full subcategory whose objects are the $u_{\hat{x}}$ where $x \in \text{Ob}\mathcal{C}$. ‘Full’ means that
for each pair of objects $y, z$ of $D$ we have $\text{Hom}_D(y, z) = \text{Hom}_C(y, z)$. Define $F(x) = \hat{x}$, and for each morphism $\alpha : x \to y$ define $F(\alpha) : F(x) \to F(y)$ to be $i_y \alpha i_x^{-1}$.

(a) Show that $F$ is a functor.

(b) Show that $F$ and the inclusion functor $\text{inc} : D \to C$ are inverse equivalences of categories $D \simeq C$. (It will help to assume that when $x = u_z$, the chosen isomorphism is the identity $1_x$.)

(c) Deduce that the category Set of finite sets is equivalent to the category with objects $\mathbb{N} := \{0, 1, 2, \ldots\}$ and where $\text{Hom}(n, m)$ is the set of all mappings of sets from $\mathbf{n} := \{1, \ldots, n\}$ to $\mathbf{m} := \{1, \ldots, m\}$. We take $0 = \emptyset$.

(d) Deduce also the following: let $K$ be a field. Show that the category Vec of finite dimensional vectors spaces over $K$ is equivalent to the category $C$ with objects $\mathbb{N} := \{0, 1, 2, \ldots\}$, where $\text{Hom}_C(n, m)$ is the set $M_{m,n}(K)$ of $m \times n$ matrices with entries in $K$, and where composition of morphisms is matrix multiplication. In case $m$ or $n$ is zero, give a definition of $\text{Hom}_C(n, m)$ that will make this question make sense.

11. Let $C$ be a small category. A self-equivalence of $C$ is an equivalence of categories $F : C \to C$. Show that the set of natural isomorphism classes of self equivalences of $C$ is a group, with multiplication induced by composition of functors.