The dimension of affine rings

Affine rings = finitely generated algebras over a field k.

It's nice to know

Theorem.
If k is a field then $\dim k[x_1, \dots, x_r] = r$

Noether Normalization

If A is a graded algebra finitely generated over k then the degree of the Hilbert polynomial is $\dim A - 1$

Lemma 13.2. in Eisenbud.

Let k be a field and f in T = k[x_1, … ,x_r] a non-constant polynomial.

There are elements x′_1, … , x′_{r-1} in T so that

T is a finitely generated module over the k-subalgebra generated by x′_1, … , x′_{r-1} and f. T is integral over $S =$ subalg. gen'd by $x'_1, \ldots, x'_{r-1}, f$

The x′_i can be chosen in various ways. We will show that we can choose x′_i of the form x_i - x_r^{e^i} for any sufficiently large e. $\in \mathbb{Z}$

If f is homogeneous, they can be chosen (in a different way) homogeneous.

Proof. Write f as a polynomial in $x'_1, \ldots, x'_{r-1}, x_r$.

This means: where we see x_i we write $x'_i + x_r^{e^i}$

A monomial $x_1^{a_1} \ldots x_r^{a_r}$ becomes

$$(x'_1 + x_r^{e^1})^{a_1} \cdots (x'_{r-1} + x_r^{e^{r-1}})^{a_{r-1}} x_r^{a_r}$$

$$= x_1'^{a_1} \cdots x_{r-1}'^{a_{r-1}} x_r^{a_r} + \cdots$$

$$+ x_r^{\underbrace{a_1 e^1 + \cdots a_{r-1} e^{r-1} + a_r}_{=d}}$$

If $e \gg 0$ the last term has highest degree x_r^d.

For different monomials, the highest degrees d are distinct (the a_i are the digits in the e-adic expansion of d).

No cancellation occurs between them.

The very highest degree is a power of x_r

x_r is a root of a monic polynomial over $k[x'_1, \ldots, x'_{r-1}, f] = S$ $\quad f(x_r) - f$

Thus T is generated over S by $1, x_r, x_r^2, \ldots, x_r^{d-1}$. $\square$

Pre-class Warm-up

Is the following statement obvious?

Let A be a commutative ring and let B be a homomorphism image of A.
Then dim A $\leq$ dim B.

A Yes, it is obvious

B No, it is not obvious.

We have the understanding that if a statement is not true, then it is not obvious.

Today: we will show

$\dim k[x,y]/(x^2-y^3) = 1$.

$\dim k[x_1,\dots,x_d] = d$.

Theorem 13.1
Let k be a field. Then
dim k[x_1, … ,x_r] = r.

Proof. Write $T = k[x_1, \ldots, x_r]$.
Induction on r.
$r=0$: $T = k$ has dimension 0.
Suppose $r > 0$ and result holds for $r-1$.
$0 \subset (x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, \ldots, x_r)$
is a chain of prime ideals so
$\dim T \leq r$.
Let $0 \subset P_1 \subset \cdots \subset P_m$ be a chain of prime ideals. Let $0 \neq f \in P_1$
We can find $x_1', \ldots, x_{r-1}'$ so that
T is integral over $S := k[x_1', \ldots, x_{r-1}', f]$

By 'incomparability'
$0 \subset S \cap P_1 \subset \cdots \subset S \cap P_m$ is a chain of distinct prime ideals.
Factor out (f) to get a chain of length $m-1$ in $S/(f)$:

$$\frac{S \cap P_1}{(f)} \subset \cdots \subset \frac{S \cap P_m}{(f)}$$

$S/(f)$ is an image of $k[x_1', \ldots, x_{r-1}']$ which has dimension $\leq r-1$, by induction (image of a polynomial ring with $r-1$ variables).
Hence $m-1 \leq r-1$, $m \leq r$. $\square$

Corollary. Any k-algebra that can be generated by r elements has dimension ≤ r.

Example.

Proof. In $k[x,y]$ take $f = x^2 - y^3$

We can find x' so that

$k[x,y]$ is integral over $k[x', f]$

$A = k[x,y]/(x^2-y^3)$ is

integral over $k[x', f]/(f)$

$\cancel{\cong}$ an image of $k[x']$ which has dim 1

Therefore $\dim A \leq 1$.

$\dim A \neq 0$.

$0 \subset (\bar{x}, \bar{y})$ is a chain of prime ideals.

Simplified and weaker form of Noether's

see Atiyah-Macdonald

Theorem (compare 13.3 in Eisenbud)
Let k be a field and let $A \neq 0$ be a finitely generated k-algebra (an 'affine ring over k'). There is a polynomial subring $S = k[x_1, \dots, x_d]$ such that A is integral over S.

Proof. Let $y_1, \dots, y_n$ generate A as a k-algebra.
Renumber ~~them so that~~ $y_1, \dots, y_r$ are algebraically indept over k, each of $y_{r+1}, \dots, y_n$ is algebraic over $k[y_1, \dots, y_r]$.
Induction on n. $n = r$: done.
Suppose $n > r$, and result holds for $n-1$ generators.

y_n is algebraic over $k[y_1, \dots, y_{n-1}]$
so $\exists$ non-zero polynomial f so that
$f(y_1, \dots, y_{n-1}, y_n) = 0$.
By lemma 13.2 $\exists$ $x_1', \dots, x_{n-1}'$ in A so that A is integral over
$k[x_1', \dots, x_{n-1}', f] \subset k[x_1', \dots, x_{n-1}']$
By induction $k[x_1', \dots, x_{n-1}']$ is integral over a polynomial ring S.
Integral extensions remain integral. $\square$

Is it obvious?: $k[x_1, \dots, x_d]$ is a polynomial ring <=> $x_1, \dots, x_d$ are algebraically independent over k?

Theorem (compare 13.3 in Eisenbud)
Let k be a field and let $A \neq 0$ be a finitely generated k-algebra (an 'affine ring over k').
There is a polynomial subring
$S = k[x_1, \dots, x_d]$ such that A is integral over S.

Furthermore, if $I_1 \subset I_2 \subset \cdots I_m$
is a chain of ideals of A with $\dim A/I_j = d_j$
and $d_1 > d_2 > \dots > d_m > 0$
then S can be chosen so that

$$I_j \cap S = (x_{d_j}, \dots, x_d) .$$

If the ideals are homogeneous, the x_j can be chosen to be homogeneous.

Theorem. If A is an affine domain over a field k, then $\dim A = \text{tr.deg.}_k R$.

Proof. Let $S = k[x_1, \dots ,x_d]$ be the polynomial subring of A over which A is integral. Then
tr.deg. $S = d$
and the field of fractions of A is a finite degree extension of S, so tr.deg. $A = d$ also.