How would you solve the following equations?

The options are:

a. integrate \( y' = \text{function of } x \)
b. separate the variables
c. first order linear equation
d. make a special substitution
e. homogeneous equation
f. Bernoulli equation
g. exact equation
h. reduce the order

1. \( xy' = y + \frac{y^2}{x} \)
2. \( x \frac{dy}{dx} + 6y = 3x y^{4/3} \)
3. \( y' = \sqrt{x+y+1} \)
4. \( y y'' = (y')^2 \)
5. \( xy'' + y' = 4x \) \( \text{If } e^{\int \frac{1}{x} dx} \)
6. \( 2xy \frac{dy}{dx} + y^2 = 10x \) \( \frac{d}{dx}(xy^2) = 10x \)
7. \( xy^2 + 3y^2 - x^2 y' = 0 \)
8. \( 2xy + x^2 y' = y^2 \) \( \text{If } e^{\int \frac{3}{x} dx} \)
9. \( x^3 + 3y - xy' = 0 \) \( \text{If } e^{\int \frac{3}{x} dx} \)

\( v = \frac{y}{x} \) \( x \frac{dv}{dx} = v^2 \)
\( v = y' - \frac{y}{x} \)
\( v = \frac{y+1}{2v} \) \( \text{separate variables} \)
\( v = y' \) \( y v \frac{dv}{dy} = v^2 \) \( \text{separate} \)
• Calculate \( e^{At} \) where \( A = \begin{bmatrix} 2 & 1 \\ i & 2 \end{bmatrix} \)

(Note that \( A \) has eigenvalues 1, 3 with eigenvectors \( \begin{bmatrix} 1 \\ i \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \))

Also \( \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \)

a. \( \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix} \)

b. \( \begin{bmatrix} e^t & te^{3t} \\ -e^t & e^{3t} \end{bmatrix} \)

c. \( \frac{1}{2} \begin{bmatrix} e^t + e^{3t} & e^t + e^{3t} \\ -e^t + 3t & e^t + e^{3t} \end{bmatrix} \)

• Classify all the critical points of \( x' = Ax \) (as node (proper, improper), center, saddle point, spiral point, stable, unstable etc etc).

• How would you find a particular solution to \( x' = Ax + \begin{bmatrix} e^t \\ 0 \end{bmatrix} \)
How would you find a particular solution to
\[ x' = Ax + \begin{bmatrix} e^t \\ 0 \end{bmatrix} \]
where \( A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \)?

a. Try a solution \( \begin{bmatrix} a \\ b \end{bmatrix} e^t + \begin{bmatrix} c \\ d \end{bmatrix} e^{3t} \)

b. Try a solution \( A \begin{bmatrix} a \\ -1 \end{bmatrix} e^t + B \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \)

c. Try a solution \( \begin{bmatrix} a \\ b \end{bmatrix} e^t + \begin{bmatrix} c \\ d \end{bmatrix} e^t \)

d. something else.
How would you calculate $e^{At}$ where

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$$

a. substitute $At$ into the power series for $e^u$

b. solve the differential equation $x' = Ax$ and do a calculation with the fundamental matrix

c. break apart $A$ into two pieces, and substitute those into the power series for $e^u$

d. something else.

Same question when $A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$

Same question when $A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$
Section 5 page 325 question 26 +

Which of the following is the appropriate form of a solution of the non-homogeneous equation

\[ x'' - 6x' + 13x = t e^{3t} \sin(2t) \]

Note that the characteristic polynomial has roots 3+2i and 3-2i

a. \( a t e^{3t} \sin 2t + b t e^{3t} \cos 2t \)

b. \( a t e^{3t} \sin 2t + b t e^{3t} \cos 2t + c e^{3t} \sin 2t - d e^{3t} \cos 2t \)

c. \( a t e^{3t} \sin 2t + b t e^{3t} \cos 2t \)

d. None of the above.

e. Formula a + Formula c.

What about \( x'' - 6x' + 13x = t \sin(2t) \)

a. \( a t e^{3t} \sin 2t + b t e^{3t} \cos 2t \)

b. \( a t \sin 2t + b t \cos 2t + c \sin 2t + d \cos 2t \)

Exam in Murphy 130

For things you don't need to know only go by what I have told you. You don't need to know.

Please check the grading if you got a low score on any question (in Exam 3)
Linear algebra

True/false for a linear equation $Ax = b$ where $A$ is a $2 \times 2$ matrix?

• If $A$ is invertible, then the above always has a unique solution for any $b$.
• If the rank of $A$ is 1, we can always find a vector $b$ in $\mathbb{R}^2$ so that $Ax = b$ does not have a solution.
• Suppose $b = 0$. If the rank of $A$ is 2, we can always find a solution $x \neq 0$.
• Suppose $v_1$ and $v_2$ are both solutions. Then $3v_1 + 2v_2$ is also a solution.
• If the column vectors of $A$ are linearly independent, then $A$ has an inverse.
• The $2 \times 2$ zero matrix is the only $2 \times 2$ matrix whose null space is 2-dimensional.

Questions not about a particular equation $Ax = b$. True/False?

- We can find a $2 \times 3$ matrix whose null space is the zero vector space. $0 + \leq 2 \neq 3$
- We can find a $2 \times 3$ matrix whose null space is the space spanned by the vector $(1,2)$. Null space vectors have length $3$.
- We can find a $2 \times 3$ matrix whose null space is the space spanned by the vector $(1,2,0)$ and whose column space is the space spanned by $(2,1)$. $\dim \text{nullsp} + \text{rank} \neq 3$
- We can find a $2 \times 3$ matrix whose null space is the space spanned by the vector $(1,2,0)$ and whose column space has dimension 2.
- We can find a $2 \times 3$ matrix whose column space has dimension 3.
True/false
Let $A$ be an $m \times n$ matrix with $m < n$.

1. $Ax = b$ always has a solution.
2. $Ax = 0$ always has a solution.
3. $Ax = 0$ always has infinitely many solutions.
4. $Ax = 0$ always has a non-zero solution.
5. The columns of $A$ are necessarily dependent.
6. The rows of $A$ are necessarily independent.

Suppose now just that $m \leq n$

1. If $Ax = b$ has a solution for every $b$ then the columns of $A$ are independent.
2. If $Ax = b$ has a solution for every $b$ then the columns of $A$ span $\mathbb{R}^m$.
3. If $Ax = b$ has a solution for every $b$ then the rank of $A$ is $n$. 
A car starts from rest and its engine accelerates it at 10 ft/sec². Also, air resistance provides 0.1 ft/sec of deceleration for each ft/sec of the car’s velocity.

(a) Find the car’s maximum possible velocity.
(b) How long does it take to attain 90% of the limiting velocity, and how far does it travel in doing this?

Start with: What equation is relevant for this problem?

a. Let \( x(t) = \text{position at time } t \)
   \[ x'' + 10x' + \frac{1}{10} = 0 \quad x'' = 10 - 0.1x' \]

b. \[ x' = \frac{x}{10} - 10 = 0 \]

c. \[ \frac{dv}{dt} + \frac{v}{10} - 10 = 0 \quad v = x' \]

d. None of the above

**Question (a):** Solve \( x'' = 0 \)

\[ a. \ 10 \quad b. \ 90 \quad c. \ 100 \quad d. \ 110 \quad e. \ 1000 \]

Solve \[ \frac{dv}{dt} + \frac{v}{10} = 10 \]

Separate variables, etc.

Get \( v = \ldots \). Solve \( v = 90 \) to get \( t \).

How far? \[ x = \int v \, dt \]

and put in \( t \).
A tank contains 60 gallons of pure water. Brine with concentration 1 lb salt per gallon enters at 2 gal/min. Perfectly mixed solution leaves at 3 gal/min. Thus the tank is empty after 1 hour.

(a) Find the amount of salt in the tank after \( t \) minutes,
(b) What is the maximum amount of salt ever in the tank?

\[
\begin{align*}
\text{Let } x(t) \text{ lb salt be in the tank. } \\
\frac{dx}{dt} &= \frac{2}{60-t} - \frac{3x}{60-t} \\
\text{concentration} \\
\end{align*}
\]

\[ \text{How do we go about solving this equation? Integrating factor etc.} \]

What equation is appropriate for solving this problem?

\[ \begin{array}{ll}
\text{a} & \frac{dx}{dt} = 2 - \frac{3x}{60-t} \\
\text{b} & \frac{dx}{dt} = \frac{2}{60-t} - \frac{3x}{60-t} \\
\text{c} & \frac{dx}{dt} = \frac{2-3x}{60-t} \\
\text{d} & \text{None of the above} \\
\end{array} \]

What is the volume of liquid in the tank at time \( t \)?

\[ \begin{array}{ll}
\text{a} & t+60 \\
\text{b} & 60t < 60-t \\
\text{c} & \text{None of the above} \\
\end{array} \]
A tank contains 60 gallons of pure water. Brine with concentration 1 lb salt per gallon enters at 2 gal/min. Perfectly mixed solution leaves at 3 gal/min. Thus the tank is empty after 1 hour.

(a) Find the amount of salt in the tank after $t$ minutes,
(b) What is the maximum amount of salt ever in the tank?

Solution:

Let $x(t)$ be the amount of salt in the tank at time $t$. The volume of liquid in the tank at time $t$ is $60-t$. The concentration of salt at time $t$ is $\frac{x(t)}{60-t}$.

We get \[
\frac{dx}{dt} = 2 - \frac{3x}{60-t}.
\]

Divide by $60-t$:

\[
\frac{dx}{dt} + \frac{3x}{60-t} = 2.
\]

Integrate factor:

\[
I.F. = e^{\int \frac{3}{60-t} dt} = e^{\ln(60-t)} = \frac{1}{60-t}.
\]

Multiply both sides by $\frac{1}{60-t}$:

\[
\frac{1}{(60-t)^3} \frac{dx}{dt} + \frac{3x}{(60-t)^4} = \frac{2}{(60-t)^3}.
\]

Taking the derivative:

\[
\frac{d}{dt} \left( \frac{x}{(60-t)^3} \right) = \frac{2}{(60-t)^3}.
\]

The concentration of salt at time $t$ is $\frac{x(t)}{60-t}$. We get $\frac{dx}{dt} = 2 - \frac{3x}{60-t}$. We integrate factor $I.F. = e^{\int \frac{3}{60-t} dt} = e^{\ln(60-t)} = \frac{1}{60-t}$. Multiply both sides by $\frac{1}{60-t}$.

\[
\frac{1}{(60-t)^3} \frac{dx}{dt} + \frac{3x}{(60-t)^4} = \frac{2}{(60-t)^3}.
\]

Taking the derivative:

\[
\frac{d}{dt} \left( \frac{x}{(60-t)^3} \right) = \frac{2}{(60-t)^3}.
\]
\[
\frac{d}{dt} \left( \frac{x}{(60-t)^3} \right) = \frac{2}{(60-t)^3}
\]
\[
\frac{x}{(60-t)^3} = \frac{1}{(60-t)^2} + C
\]
\[
x = 60-t + C(60-t)^3
\]
\[
x(0) = 0 = 60 + C(60)^3
\]
\[
C = -\frac{1}{60^2}
\]
\[
x = 60-t - \frac{(60-t)^3}{60^2}
\]

To find the maximum, solve
\[
\frac{dx}{dt} = 0
\]
Like section 3.2 questions 23-26:

Determine for what values of $k$ and $c$ the system has (a) a unique solution (b) no solution (c) infinitely many solutions

\[
\begin{align*}
3x + 2y &= 1 \\
6x + cy &= k 
\end{align*}
\]
Like section 9.1 13-20

Identity whether the critical point (0,0) is stable, asymptotically stable or unstable. From aspects of the solutions, identify it as a node, a saddle point, a center or a spiral point.

20. $\frac{dx}{dt} = y, \frac{dy}{dt} = -5x - 4y$

\[ \lambda = -2 \pm i \]

Asymptotically stable spiral point.

This gives solutions $e^{-2t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} e^{2t}$.

We do not need to distinguish proper and improper nodes.