Can you remember what Newton’s law of cooling says? Does it say:

a. \( \frac{dT}{dt} = k(t - A) \)

b. \( \frac{dT}{dt} = k(T - A) \)

c. \( \frac{dT}{dt} = k(T - A) \)

d. \( \frac{dT}{dt} = -k(T - A) \)

e. None of the above
Section 1.5: Linear first order differential equations

We learn:
• what does a linear equation look like?
• How to solve them
• How to do questions about tanks of brine.

We don’t need:
• The theoretical statement of Theorem 1 on page 50 about the existence and uniqueness of solutions

A linear differential equation is a linear combination of derivatives by functions of $x$, like:

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P(x)y' + P_0(x)y = Q(x)$$

A first order linear d.e. has the form:

$$P_1(x)y' + P_0(x)y = Q(x)$$

and we can write it:

$$y' + P(x)y = Q(x)$$

Question: which of the following are linear d.e.’s?

a. $y' = x-y$  Yes  No

b. $y y' + e^x = x^{15}$  Yes  No

c. $y' + ye^x = x^{15}$  Yes  No
How to solve \( \frac{dy}{dx} + P(x) y = Q(x) \) ?

We multiply both sides by \( e^{\int P(x) \, dx} \) to get

\[
\frac{d}{dx} (e^y) = e^y y' + e^y P y = e^y Q
\]

Integrate both sides with respect to \( x \) to get

\[
xe^y = \int xe^y \, dx = e^y \int e^y \, dx
\]

Divide by \( e^y \) to get

\[
y = x - 1 + Ce^{-x}
\]

Question: Solve \( \frac{dy}{dx} = x - y \)

Solution

\[
y' + y = x \quad \text{so} \quad P = 1, \quad \int P = x,
\]

I.F. = \( e^x \)

\[
e^x y' + e^x y = xe^x
\]

\[
\frac{d}{dx} (e^x y) = xe^x
\]

\[
e^x y = \int xe^x \, dx
\]

\[
y = x - 1 + Ce^{-x}
\]
Page 56 question 24.
Solve \((x^2 + 4) y' + 3xy = x, \ y(0) = 1.\)

Solution: \[ y' + \frac{3x}{x^2+4} y = \frac{x}{x^2+4} \]

The Integrating Factor is:

\[ e^{\int \frac{3x}{x^2+4} \ dx} = e^{\ln (x^2+4)^{3/2}} = (x^2+4)^{3/2} \]

Multiply to get

\[ (x^2+4)^{3/2} y' + 3x (x^2+4)^{1/2} y = x (x^2+4)^{1/2} \]

\[ \frac{d}{dx} \left[ (x^2+4)^{3/2} y \right] = x (x^2+4)^{1/2} \]

\[ (x^2+4)^{3/2} y = \frac{1}{3} (x^2+4)^{3/2} + C \]

\[ y = \frac{1}{3} + \frac{C}{(x^2+4)^{3/2}} \]

\[ 1 = \frac{1}{3} + \frac{C}{8} \quad y(0) \]

\[ C = \frac{16}{3} \]

We get \[ y = \frac{1}{3} + \frac{16}{3 (x^2+4)^{3/2}} \]
Question: Which method would you use to solve the differential equation

\[ \frac{dy}{dx} = ye^x \]

a. Separate the variables

b. Treat it as a linear first order equation

c. Do something else

\[ \text{Both work} \]
A tank contains 60 gallons of pure water. Brine with concentration 1 lb salt per gallon enters at 2 gal/min. Perfectly mixed solution leaves at 3 gal/min. Thus the tank is empty after 1 hour.

(a) Find the amount of salt in the tank after \( t \) minutes,

(b) What is the maximum amount of salt ever in the tank?

**Solution:**

Let \( x(t) \) be the amount of salt in the tank at time \( t \).

The volume of liquid in the tank at time \( t \) is \( 60 - t \).

The concentration of salt at time \( t \) is \( \frac{x(t)}{60-t} \) lb/gal.

We get

\[
\frac{dx}{dt} = 2 - \frac{3x}{60-t}
\]

\[
\frac{dx}{dt} + \frac{3x}{60-t} = 2
\]

The integrating factor (I.F.) is \( e^{\int \frac{3}{60-t} dt} = e^{3 \ln(60-t)} = \frac{1}{(60-t)^3} \)

\[
\frac{1}{(60-t)^3} \frac{dx}{dt} + \frac{3x}{(60-t)^4} = \frac{2}{(60-t)^3}
\]

\[
\frac{d}{dt} \left( \frac{x}{(60-t)^3} \right) = \frac{2}{(60-t)^3}
\]
\[
\frac{d}{dt} \frac{x}{(60-t)^3} = \frac{2}{(60-t)^3}
\]

\[
\frac{x}{(60-t)^3} = \frac{1}{(60-t)^2} + C
\]

\[
x = 60-t + C(60-t)^3
\]

\[
x(0) = 0 = 60 + C \cdot 60^3
\]

\[
C = \frac{-1}{60^2}
\]

\[
x = 60-t - \frac{(60-t)^3}{60^2}
\]

To find the maximum distance,
\[
\frac{dx}{dt} = 0
\]